Essential electromagnetism for photonic metamaterials

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Photonic metamaterials

Metamaterials: Man-made structured materials (composites) with properties different than those of the constituent media – result of the structuring (shape and size of their components). Properties mostly non-existent in natural materials

Photonic Metamaterials (electromagnetic metamaterials): metamaterials aimed to control photons (electromagnetic (EM) waves)

Essentials to study materials for photons?

- Maxwell's equations - determine the propagation of EM waves
- Electromagnetic response of a material

Outline

- Definitions of the essential electromagnetic quantities
- Maxwell's equations
- Wave equation – waves in dielectrics and metals
- Electromagnetic response (dispersive properties) of materials
- Constitutive relations – materials classifications

Linear, isotropic, homogeneous, non-magnetic materials

Basic electromagnetic quantities: Definitions

Microscopic electric and magnetic field

Let's point charge *q* moving with velocity v in fields e and b

Force on *q*:

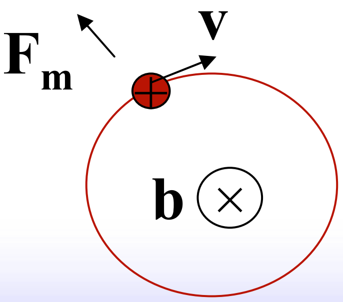
$$\mathbf{F} = \overbrace{q\,\mathbf{e}}^{\mathbf{F}_e} + \overbrace{q\,\mathbf{v}\times\mathbf{b}}^{\mathbf{F}_m}$$

Lorenz force

Microscopic electric field

Microscopic magnetic field

- **Electric field, e: Electric force ($\mathbf{F}_e$) per unit charge**
- **Magnetic field (induction), b: proportional to magnetic force ($\mathbf{F}_m$) exerted at a moving charge**



Electric force ($\mathbf{F}_e$) parallel to electric field; direction depends on charge

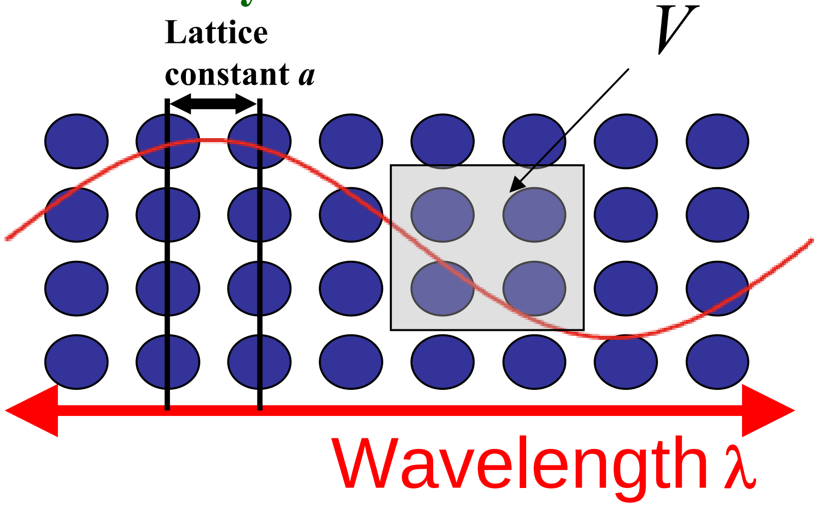
Magnetic force ($\mathbf{F}_m$) perpendicular to magnetic field; exerted at a currents

Macroscopic electric and magnetic field

Macroscopic quantities:
Averaged quantities over volume *V* such as

$$a^3 << V << \lambda^3$$

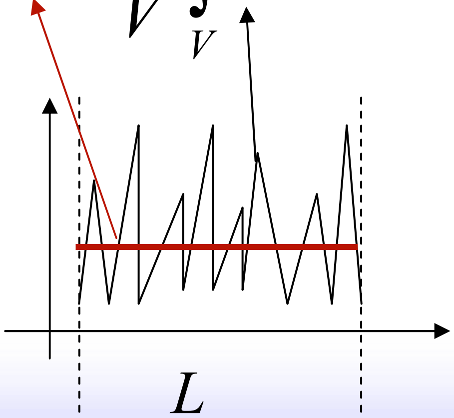
Picture by S. Linden



Electric & magnetic fields:

$$\mathbf{E} = \frac{1}{V}\int_V \mathbf{e}\, dV$$

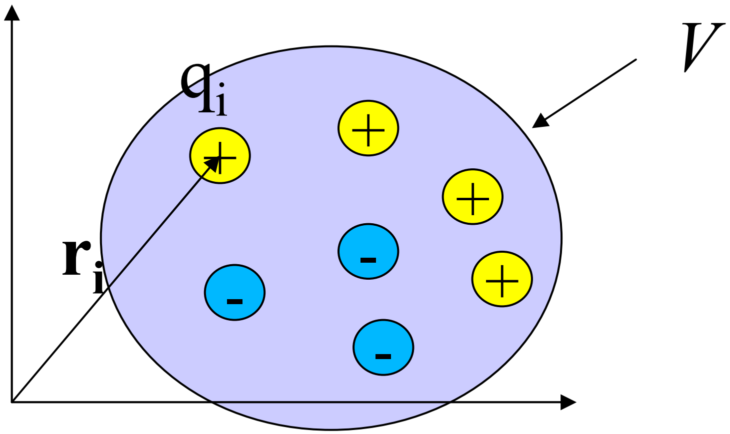
$$\mathbf{B} = \frac{1}{V}\int_V \mathbf{b}\, dV$$



Charge density and dipole moment

Charge density, ρ: Charge per unit volume

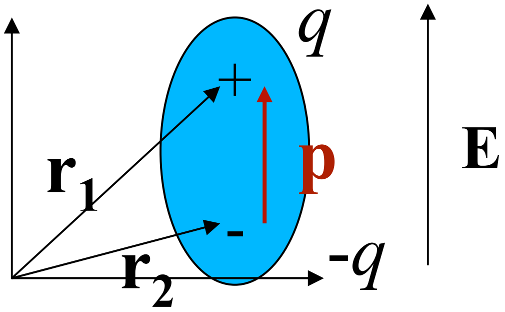
$$\rho = \frac{1}{V}\int_V \rho_m \, dV = \frac{1}{V}\sum_i q_i$$



Dipole moment, p, of a dipole (e.g. atom):

$$\mathbf{p} = q\mathbf{r}$$

Displacement of the negative in respect to the positive charge, caused by an external field



Dipole moment of a system of charges:

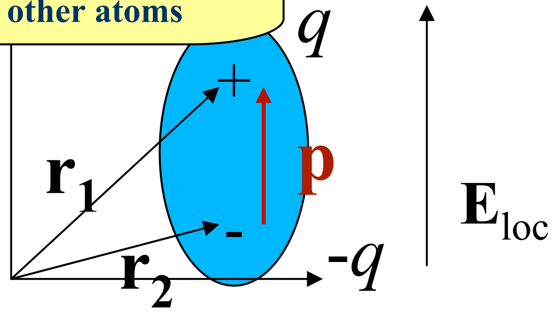
$$\mathbf{p} = \sum_i q_i \mathbf{r}_i$$

Polarizability, polarization, susceptibility

Polarizability, α :

$$\mathbf{p} = \alpha \mathbf{E}_{loc}$$

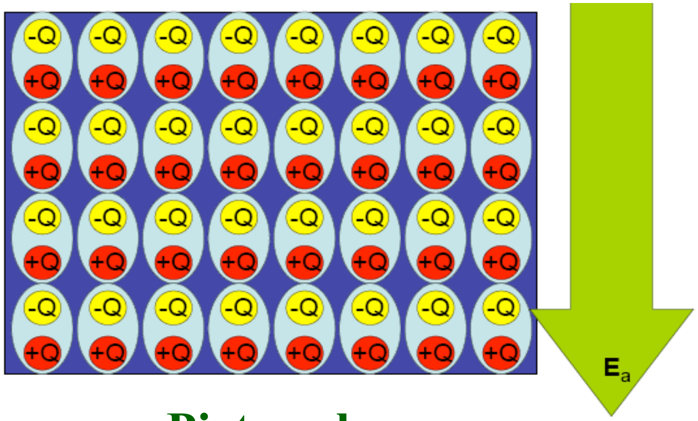
Local electric field acting on an atom, i.e. external field and field scattered by other atoms



The ability of an "atom/molecule" to be polarized

Polarization, P:
(polarization density)

$$\mathbf{P} = \frac{1}{V} \sum_i \mathbf{p}_i$$



Picture by …

Dipole moment per unit volume – how much a system of dipoles is polarized (on the average)

Electric susceptibility, χ :

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \text{ Farads/m}$$

How susceptible is a system to an overall charge displacement

Current density - magnetic dipole moment - magnetization

Local current density, j :

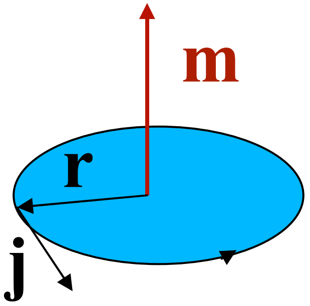
$$\mathbf{j} = q\mathbf{v}$$

Current density, J : Current per unit area

$$\mathbf{J} = \rho\mathbf{v} \ \left(= \frac{1}{V}\sum_i q_i \mathbf{v}_i\right)$$

Magnetic dipole moment, m:

$$\mathbf{m} = \frac{1}{2}\mathbf{r} \times \mathbf{j}$$

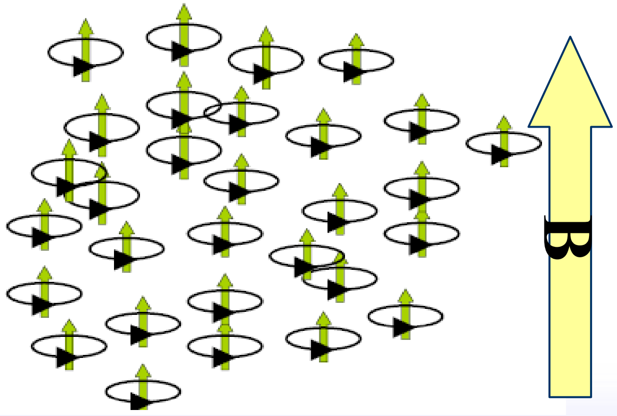


Re-orientation of an atom as to be aligned with an external magnetic field

Magnetization, M:

$$\mathbf{M} = \frac{1}{V}\sum_i \mathbf{m}_i$$

On the average reorientation of the atoms parallel to the magnetic field – strength of the interaction with B



Displacement field and magnetic field

Displacement field, D:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Magnetic field, H:

$$\mathbf{H} = \frac{1}{\mu_0}(\mathbf{B} - \mathbf{M})$$

$$\Downarrow$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$$

Fields with sources only external (or free) charges and currents

Note: **The actual total, measurable fields are E and B**

Dielectric function, Conductivity

Dielectric function/permittivity, ε :

$$\boxed{\mathbf{D} = \varepsilon \mathbf{E}}$$

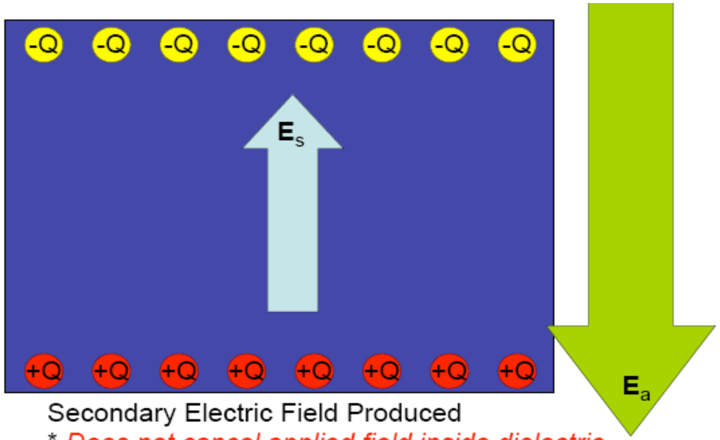
Relative permittivity, ε_r :

$$\boxed{\varepsilon_r = \varepsilon / \varepsilon_0}$$

How much the total field is reduced compared to applied field

$$\boxed{\mathbf{E}_{total} = \mathbf{E}_{applied} / \varepsilon_r}$$

$$\varepsilon = \varepsilon_0 (1 + \chi)$$



$E_s = E_{induced}$

Secondary Electric Field Produced

* *Does not cancel applied field inside dielectric*

Conductivity, σ:

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

The ability of a material to conduct an electric current

- Current is caused by an ***E***-field exerting forces on charge carriers
- Current density ***J*** depends linearly on ***E***
- Current density ***J*** depends linearly on **σ**

Resistivity, ρ:

$$\boxed{\rho = \frac{1}{\sigma}}$$

Magnetic susceptibility, permeability

Magnetic susceptibility, χ_m : $$\mathbf{M} = \mu_0 \chi_m \mathbf{B}$$

How susceptible is a system to an overall magnetization

Magnetic permeability, μ: $$\mathbf{B} = \mu \mathbf{H}$$

$\mu_0 = 4\pi \times 10^{-7}$ Henries/m

Relative permeability, μ_r: $$\mu_r = \mu / \mu_0$$

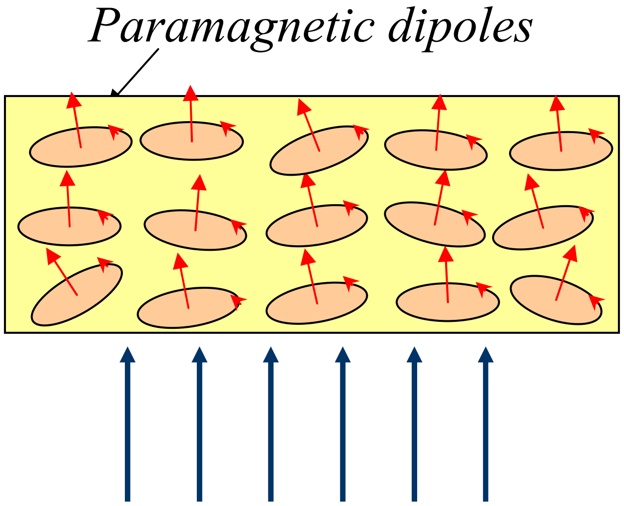
How much the total field is changed compared to applied field

$$\boxed{\mathbf{B}_{total} = \mu_r \mathbf{B}_{applied}}$$

Paramagnetic: $\mu_r > 1$

Diamagnetic: $\mu_r < 1$

(magnetic dipoles anti-allign)



$$\mathbf{B}_{applied} = \mu_0 \mathbf{H}_{applied}$$

Electric of an electric and magnetic dipole

Energy of a dipole in an external E field: $U_d = -\mathbf{p} \cdot \mathbf{E}$

Energy of a magnetic dipole in an external B field: $U_m = -\mathbf{m} \cdot \mathbf{B}$

Charge and current density in a medium

Total charge density:

$$\rho_{tot} = \rho_{free} + \rho_{bound} + \rho_{external}$$

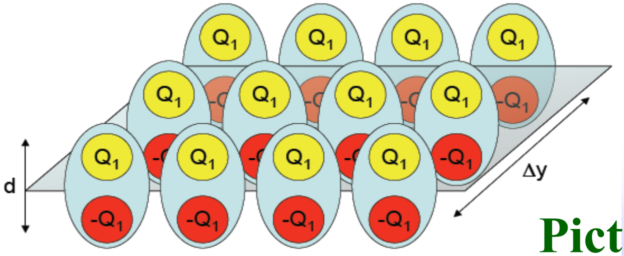
Free electrons

Bound electrons

external (source) charges

Total current density:

$$\mathbf{J}_{tot} = \mathbf{J}_{free} + \mathbf{J}_{bound} + \mathbf{J}_{external}$$



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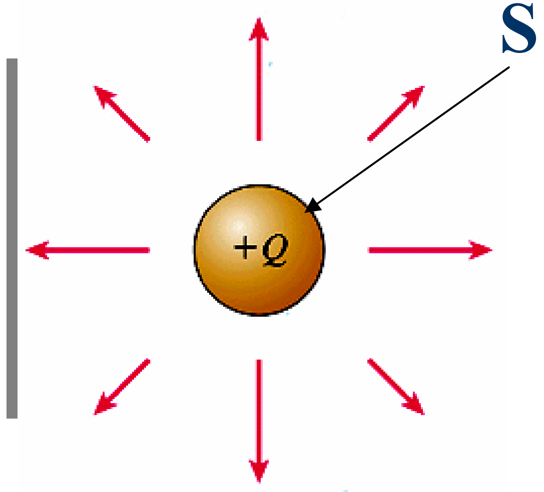
Maxwell’s equations in matter

Divergence and curl of a vector field

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{F}\, dV$$ Gauss (divergence) theorem

→ Divergence definition:

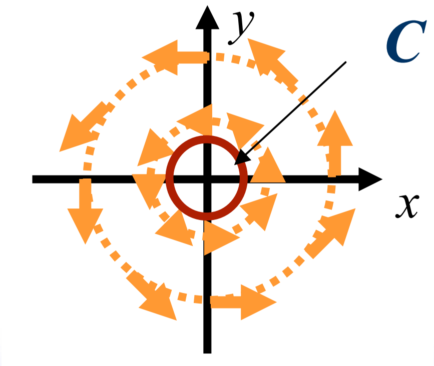
$$\nabla\cdot\mathbf{F} = \lim_{V\to 0}\frac{1}{V}\int_S \mathbf{F}\cdot d\mathbf{S}$$



Large divergence: large normal to *S* components - "strong" sources

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{F})\, d\mathbf{S}$$ Stokes theorem

→ Curl definition: $$\nabla\times\mathbf{F} = \lim_{S\to 0}\frac{1}{S}\int_C \mathbf{F}\cdot d\mathbf{l}$$



Large curl: large tangential to *C* components

Maxwell's div equations in matter

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho_{tot}}{\varepsilon_0}}$$

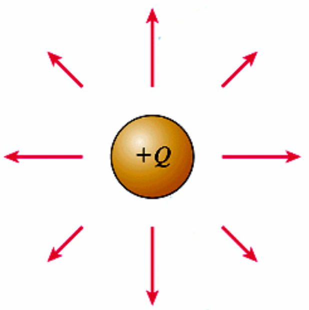
Gauss law

$$\Big\Downarrow$$

Volume integration and use of div theorem

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

- **Diverging field lines from a point indicate the presence of electric charge at that point**
- **This charge can be "detected" by surrounding the point with a surface and observing the flux through the surface**



$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

No "magnetic charges" (magnetic monopoles)

Maxwell’s curl equations in matter

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

Ampere’s law

Magnetic fields are produced by
- **currents**
- **time-varying electric fields**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

Time varying magnetic field produces an electric field

Surface integration and use of curl theorem

$$\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t}$$

Electromotive force

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$$

Magnetic flux

Maxwell's equations through D and H

Two equivalent descriptions appear in literature

Description 1: Distinction between free and bound electrons (preferred in microwaves)

Dielectric function
polarization
displacement field

describe

Bound charges' response

$$\mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Conductivity

describe

Free charges' response

$$\mathbf{J}_f = \sigma \mathbf{E}$$

Note:

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}$$

$$\nabla \cdot \mathbf{P} = -\rho$$

(Result easily from the definition equations)

Maxwell's equations through D and H

Description 2: No distinction between free and bound electrons (preferred in optics)

Dielectric function
polarization
displacement field
conductivity

describe

Total charges' response

$$\nabla \cdot \mathbf{P} = -(\rho_b + \rho_f)$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f = \sigma_{tot}\mathbf{E} = \frac{\partial \mathbf{P}}{\partial t}$$

$$\varepsilon = \varepsilon_{total} = \varepsilon_0 + i\frac{\sigma_{tot}}{\omega}$$

$$\mathbf{D} = \varepsilon_{tot}\mathbf{E}$$

Maxwell's equations through D and H

$$\boxed{\nabla\cdot\mathbf{D}=\rho}$$

Gauss law

Sources of D are only the free (if no external) charges

$$\boxed{\nabla\cdot\mathbf{B}=0}$$

$$\boxed{\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}}$$

Ampere's law

Sources of H are only the free (if no external) currents

$$\boxed{\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}}$$

Faraday's law

Description 1

$$\rho=\rho_f+\rho_{ext}$$

$$\mathbf{J}=\mathbf{J}_f+\mathbf{J}_{ext}$$

Description 2

$$\rho=\rho_{ext}$$

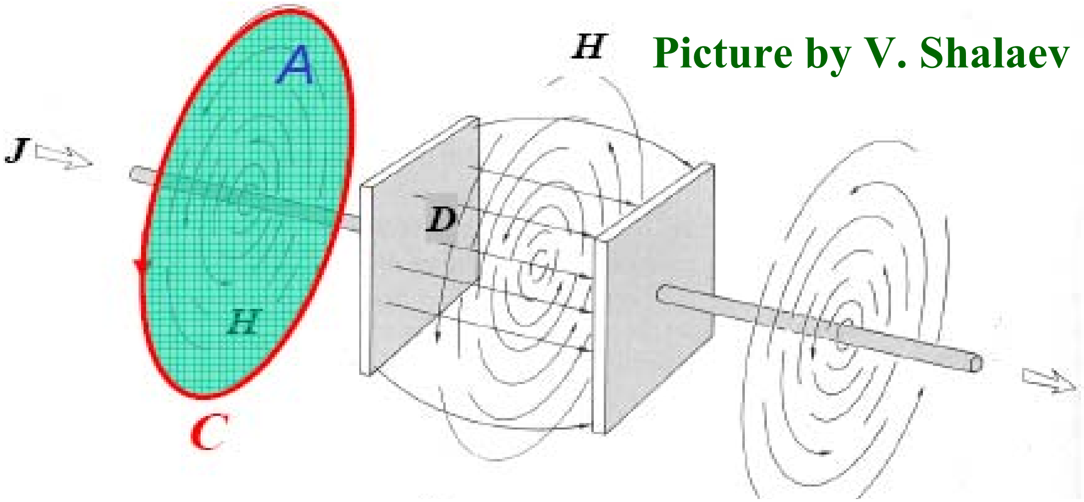
$$\mathbf{J}=\mathbf{J}_{ext}$$

$$\mathbf{J}_f\equiv\mathbf{J}_{free}$$

Ampere's law

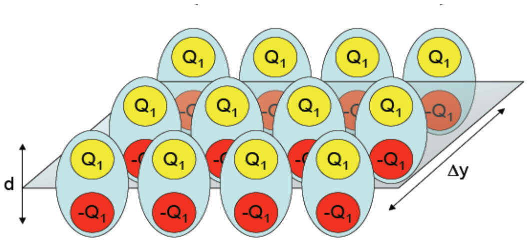
$$\boxed{\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_f}$$

Displacement current



Picture by V. Shalaev

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}$$



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In high frequencies $\mathbf{J}_\mathbf{D}$ becomes larger

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_f = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} = -i\omega\varepsilon \mathbf{E} + \sigma \mathbf{E} = -i\omega\left(\varepsilon + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

ε_{tot}

Electromagnetic Waves

From Maxwell's equations to Wave equation

Maxwell's curl equations in absence of external and free charges and currents:

$$\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \nabla\times\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times\left(\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t}\right) \Rightarrow \nabla^2\mathbf{E} = \mu\frac{\partial\nabla\times\mathbf{H}}{\partial t} = \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla\times\nabla\times\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} \qquad \nabla\times\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

Same for H

$$\boxed{\nabla^2\mathbf{E} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = 0}$$

Wave equation

$$\nabla^2 f - \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Wave velocity, υ:

$$\boxed{\upsilon = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n}} \qquad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

c = 3.0 X 10^8 m/s

Refractive index, *n*:

$$\boxed{n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}}$$

Electromagnetic waves: Definitions (1)

$$\boxed{\nabla^2 \mathbf{E} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

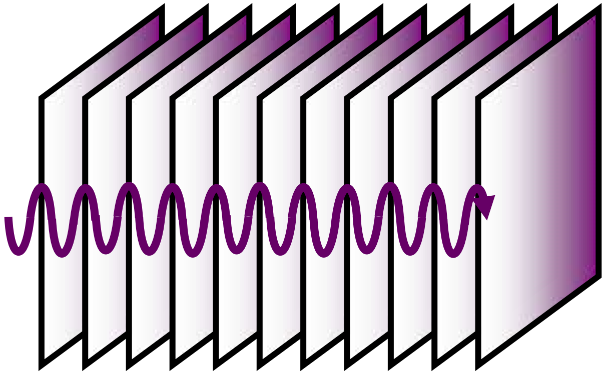
Wave equation has solutions of the form

$$\mathbf{E}(\mathbf{r}, t) = \bar{\mathbf{E}}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$$

Wave vector **Angular frequency** **Phase lag**

Solutions are called plane waves,

since the wavefonts (contours of maximum field) are planes



Picture by R. Trebino

Dispersion relation:

$$\boxed{k = \frac{\omega}{c} n}$$

Electromagnetic waves: Definitions (2)

$$\mathbf{E} = \bar{\mathbf{E}}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \varphi)$$

Dispersion relation

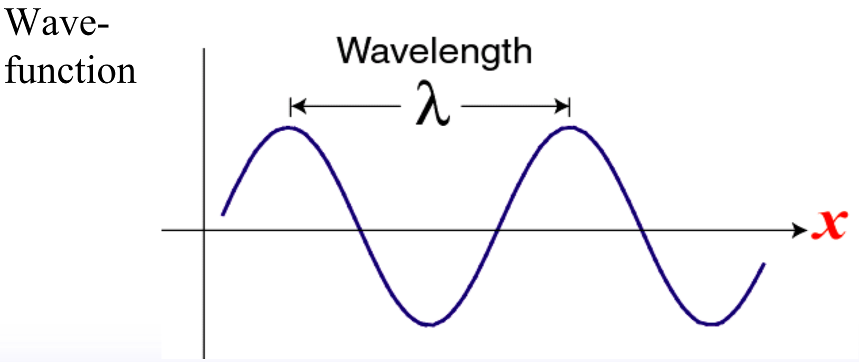
$$\boxed{k = \frac{\omega}{c} n} \longleftrightarrow \boxed{\lambda f = \frac{c}{n}}$$

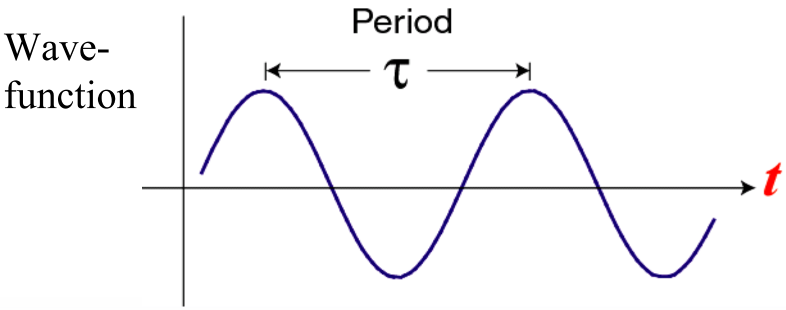
$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f = \frac{2\pi}{\tau}$$

Wavelength, **λ: distance between any successive identical parts of a wave**

Frequency f**: number of vibrations per unit time**





Pictures by R. Trebino

Electromagnetic waves and complex notation

Plane wave $\mathbf{E} = \bar{\mathbf{E}}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$

Since $\mathbf{E} = \mathrm{Re}[\bar{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\varphi)}]$ $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

we adopt complex notation, incorporating phase into a complex amplitude E_0. Thus

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Note! Physical fields are the real parts

$$\mathbf{E}_0 = \bar{\mathbf{E}}_0 e^{i\varphi}$$

Electromagnetic (EM) waves features

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

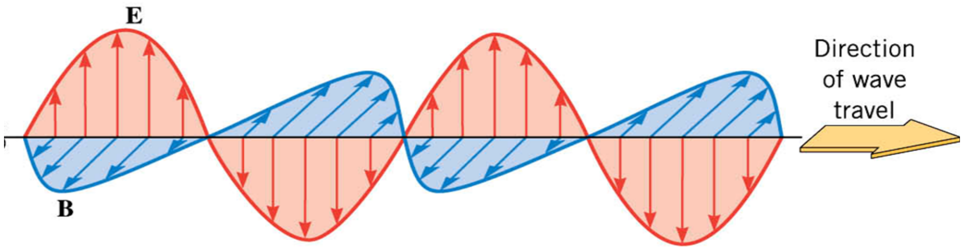
Substituting in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0 \;\Rightarrow \mathbf{k} \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0 \;\Rightarrow \mathbf{k} \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{E} = \mu\omega\mathbf{H}$$

- **EM waves are transverse**

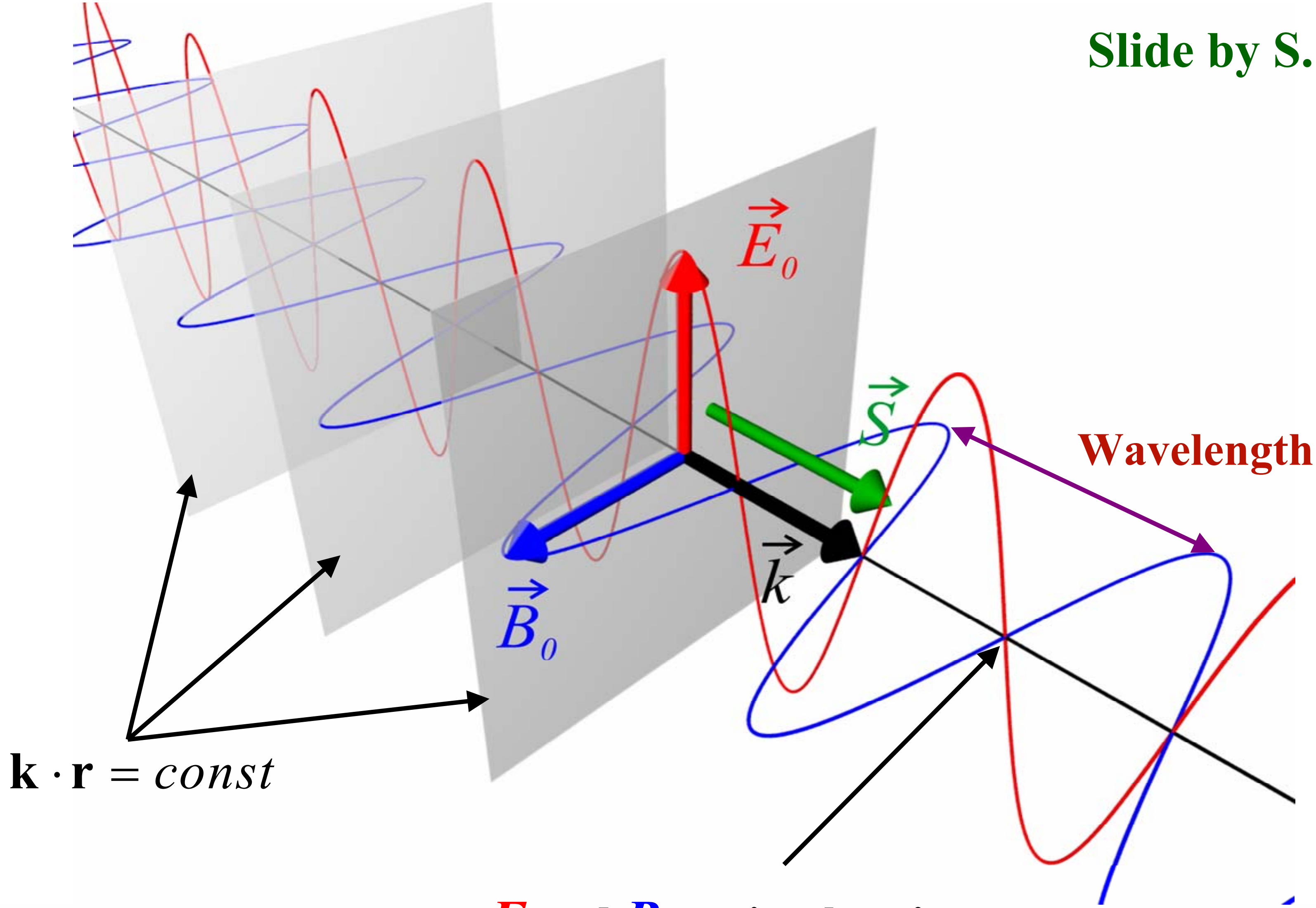


- **Magnetic and electric fields are perpendicular and in phase**

$$H_0 = \frac{k}{\mu\omega} E_0$$

Plane EM waves

Slide by S. Linden



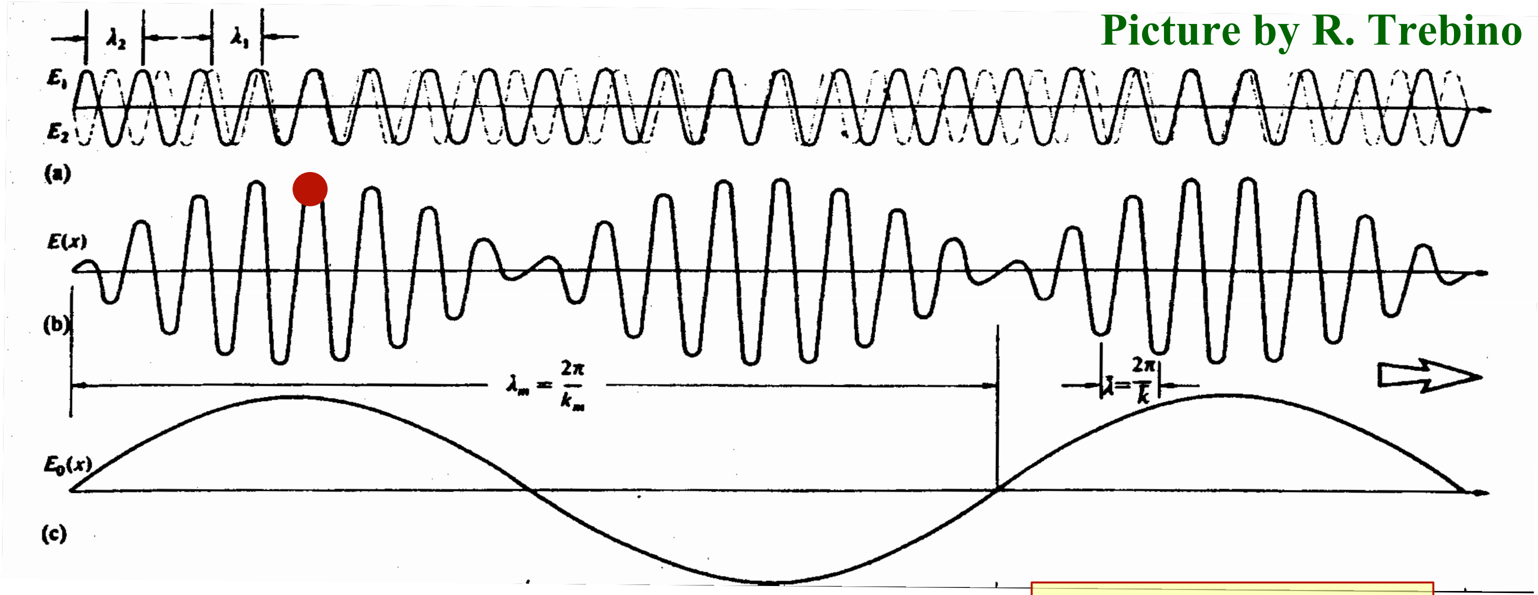
Wavelength λ

$\mathbf{k}\cdot\mathbf{r} = const$

E and *B* are in phase!

Phase and Group velocity

Adding plane waves of different wavelength produces beats:



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Individual waves

Sum

Envelope

The points of constant phase (e.g. maxima) move with the phase velocity, c_{ph}

$$kx - \omega t = const \Rightarrow \frac{d(kx - \omega t)}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$$

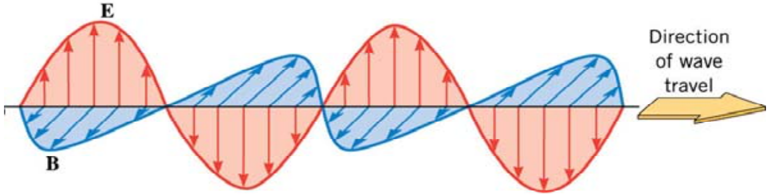
$$\boxed{c_{ph} = \frac{dx}{dt} = \frac{\omega}{k}}$$

The envelop moves with the group velocity, v_g

$$\boxed{v_g = \frac{d\omega}{dk}}$$

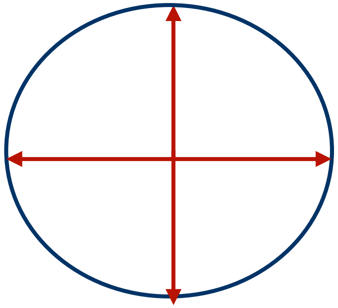
Plane wave polarization

Linearly polarized wave



the *electric field* oscillates along only one direction

Circularly polarized wave



the *electric field* amplitude writes a circle as the wave propagates

Produced by two fields of:
Equal magnitude, normal direction, phase difference 90o

$$\mathbf{E}(t) = E_0[\cos(\omega t)\hat{\mathbf{x}} + \cos(\omega t + \pi / 2)\hat{\mathbf{y}}]$$

Energy density and poynting vector

$$\mathbf{S} = \mathrm{Re}(\mathbf{E}) \times \mathrm{Re}(\mathbf{H})$$ ← **Poynting vector** **(energy flow of a wave)**

$$< \mathbf{S} >= \int_{t}^{t+\tau} \mathbf{S}\, dt = \frac{1}{2}\mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^{*}\right)$$

Time averaged over one period

The magnitude of S denotes the intensity of the electromagnetic field:

$$I = |\mathbf{S}| = \frac{1}{2}\varepsilon_0 \varepsilon c \left|\mathbf{E}_0\right|^2$$

$$w = \frac{1}{4}\mathrm{Re}\left(\mathbf{E}\cdot\mathbf{D}^{*} + \mathbf{H}\cdot\mathbf{B}^{*}\right)$$ ← **Energy density averaged over one period**

Energy per unit volume carried by an EM wave propagate in <u>non-dispersive media</u>

EM waves in metals

$$\boxed{\nabla^2 \mathbf{E} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\varepsilon = \varepsilon_0 + i\frac{\sigma}{\omega}$$

Dispersion relation: $\boxed{k = \frac{\omega}{c} n}$ **but** $\boxed{n = \sqrt{\frac{\varepsilon\mu_0}{\varepsilon_0\mu_0}} = \sqrt{1 + i\frac{\sigma}{\omega\varepsilon_0}} = n_r + in_i}$

➡ $\boxed{k = k_r + ik_i}$ ➡

In 1 dimension for simplicity

Waves of the form: $$\mathbf{E} = \mathbf{E}_0 e^{i(k_r x - \omega t)} e^{-k_i x}$$

Attenuated waves

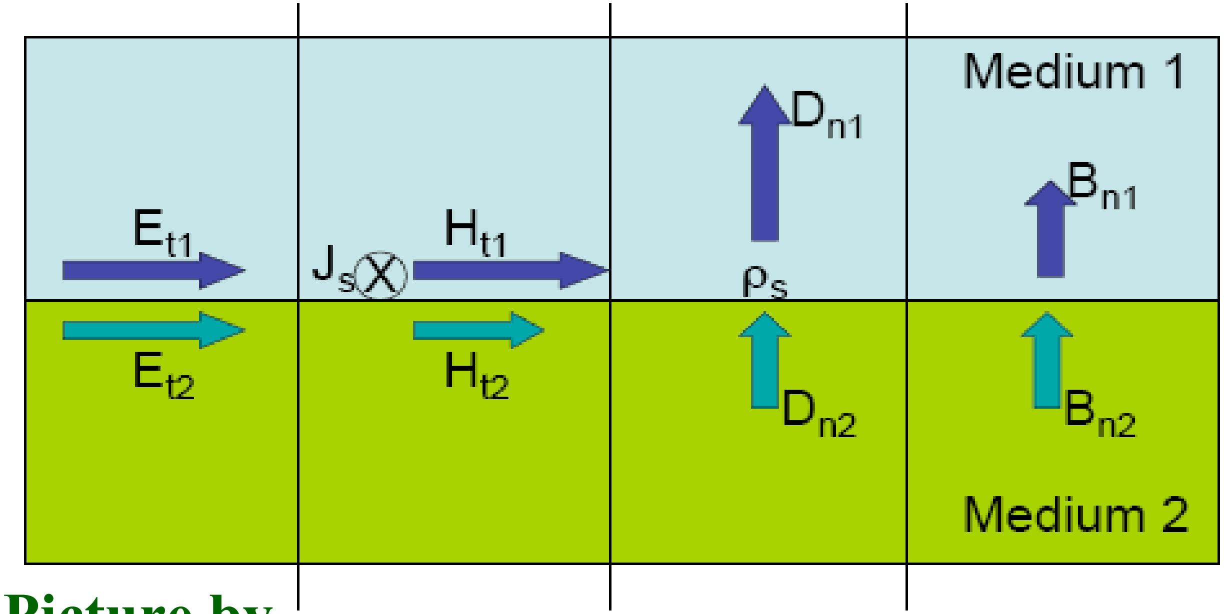
Skin depth, δ: $$\delta = \frac{1}{2k_i} \approx \frac{1}{\sqrt{2\sigma\mu_0\omega}}$$

Traveling distance required for intensity reduction e^{-1}

← **For good conductors**

Electromagnetic fields at interfaces

From Maxwell’s equations in integral form



Picture by …

E tangential
H tangential
D normal
B normal

have to be continous across charge- and current-free interfaces

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

Subscript t = tangential
Subscript n = normal

$$\frac{1}{\varepsilon_1} D_{1t} = \frac{1}{\varepsilon_2} D_{2t}$$

$$\frac{1}{\mu_2} B_{2t} = \frac{1}{\mu_1} B_{1t}$$

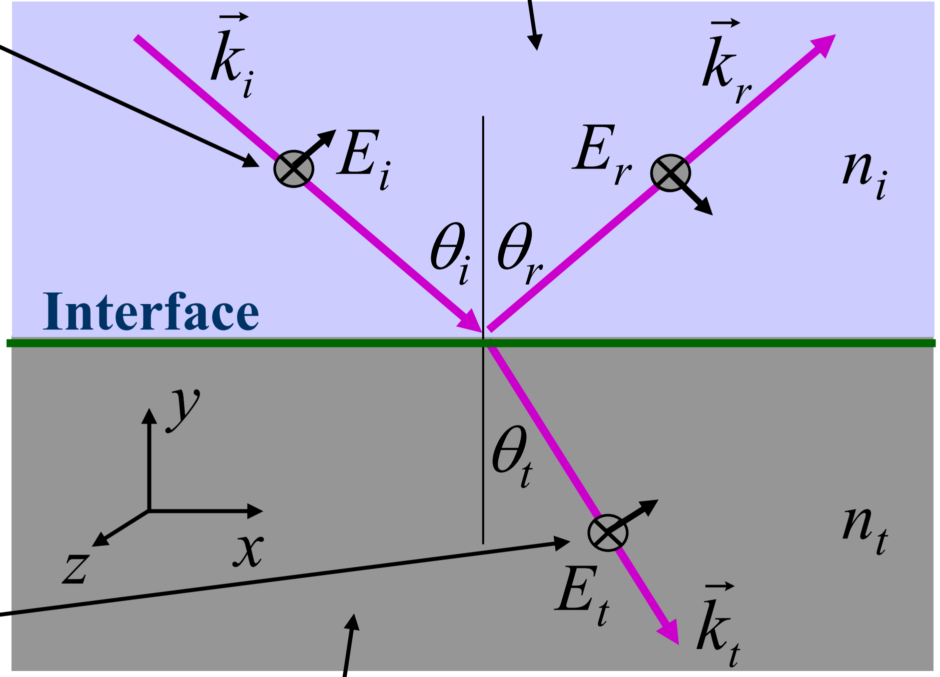
Refraction at a plane interface: Definitions

Slide by R. Trebino

Perpendicular ("S")
polarization perpendicular
to the plane of incidence

Plane of incidence is
the plane that
contains the
incident and
reflected k-vectors

Parallel ("P")
polarization lies parallel
to the plane of incidence



Incident medium

Interface

Transmitting medium

Refraction at a plane interface: Definitions

Reflection and transmission amplitudes:

$$r_\perp = E_{0r} / E_{0i}$$
$$t_\perp = E_{0t} / E_{0i}$$

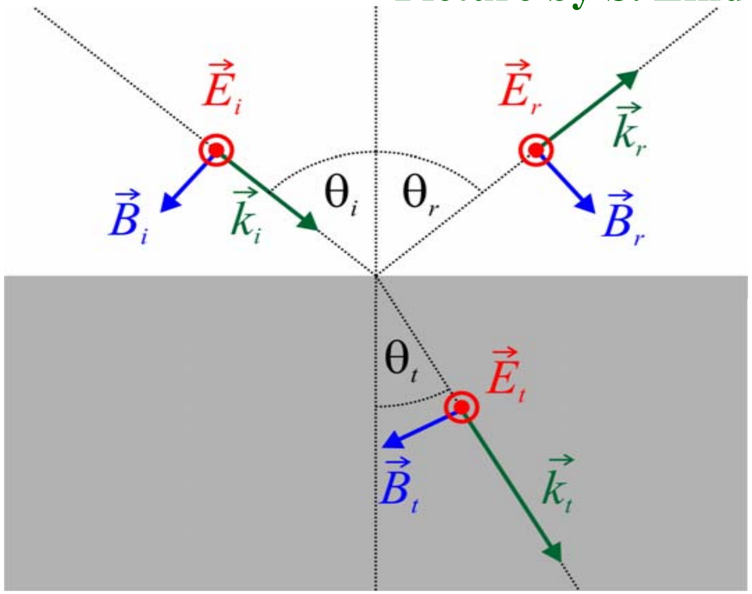
for the perpendicular polarization

$$r_{//} = E_{0r} / E_{0i}$$
$$t_{//} = E_{0t} / E_{0i}$$

for the parallel polarization

Their calculation leads to Fresnel formulas

Picture by S. Linden



E_{0i}, E_{0r}, and E_{0t} are the field complex amplitudes – calculated by applying boundary conditions

Continuity of parallel k ➡

$$\theta_i = \theta_r$$

$$\boxed{n_i \sin(\theta_i) = n_t \sin(\theta_t)}$$

⟵ **Snell's law**

Reflection and transmission coefficients

Reflection coefficient *R*

$$R = \frac{\text{Reflected poynting vector}}{\text{Incident poynting vector}} = |r|^2$$

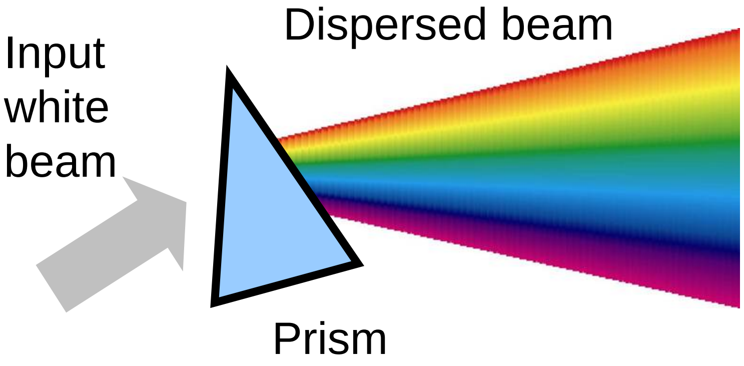
Components perpendicular to the interface

Transmission coefficient *T*

$$T = \frac{\text{Transmitted poynting vector}}{\text{Incident poynting vector}} = |t|^2 \frac{\varepsilon_2}{\varepsilon_1}$$

For lossless media *T+R=1*

Electromagnetic response (dispersive properties) of materials



Picture by R. Trebino

Dispersion is the tendency of optical properties to depend on frequency

Relations among the fields

The material properties in Maxwell's equations enter via the constitutive relations $\mathbf{D}=\varepsilon\mathbf{E},\ \ \mathbf{B}=\mu\mathbf{H}$

$\mathbf{D}(\mathbf{r},t)=\varepsilon\mathbf{E}(\mathbf{r},t)\,??$ **NO** $\mathbf{P}(\mathbf{r},t)=\varepsilon_0\chi\mathbf{E}(\mathbf{r},t)\,??$

Actual relations for time varying fields are not instantaneous

$$\mathbf{D}(t)=\int \mathrm{d}t'\,\varepsilon(t-t')\mathbf{E}(t') \qquad \mathbf{P}(t)=\varepsilon_0\int \mathrm{d}t'\,\chi(t-t')\mathbf{E}(t')$$

Polarization (and thus D) is affected by the field at previous times

Relations are neither local $\mathbf{D}(\boldsymbol{r},t)=\int \mathrm{d}t'\int \mathrm{d}^3r'\,\varepsilon(\boldsymbol{r},\boldsymbol{r}';t-t')\mathbf{E}(\boldsymbol{r}',t')$

Polarization affected by the neighborhood of point r

Relations among the fields

After Fourier transform

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\exp(-i\omega t)\,dt$$

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega) \qquad \mathbf{P}(\omega) = \varepsilon_0\chi(\omega)\mathbf{E}(\omega)$$

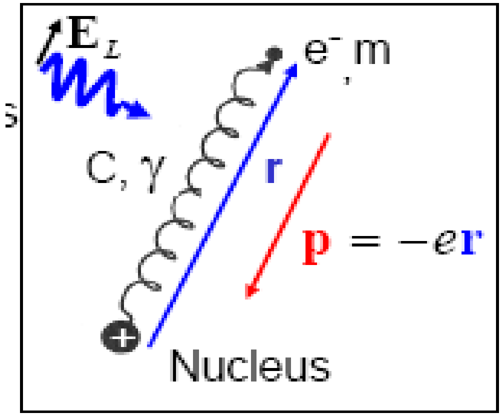
➡ **Material parameters are functions of frequency**
I.e., slow response of matter results to frequency dependent ε

For long-range interactions are also functions of wavenumber

Calculation of the material response for dielectrics

A bound electron in an electric field is treated as a forced harmonic oscillator

Picture by V. Shalaev



2nd Newton's law

$$m_e \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}_{damping} + \mathbf{F}_{restoring} + \mathbf{F}_{external}$$

$$m_e \frac{d^2\mathbf{r}}{dt^2} + m_e\gamma\frac{d\mathbf{r}}{dt} + m_e\omega_0^2\mathbf{r} = -e\tilde{\mathbf{E}}_L\exp(-i\omega t)$$

Resonance frequency

Damping factor

$$\mathbf{p} = -e\mathbf{r} = \mathbf{p}_0\exp(-i\omega t)$$

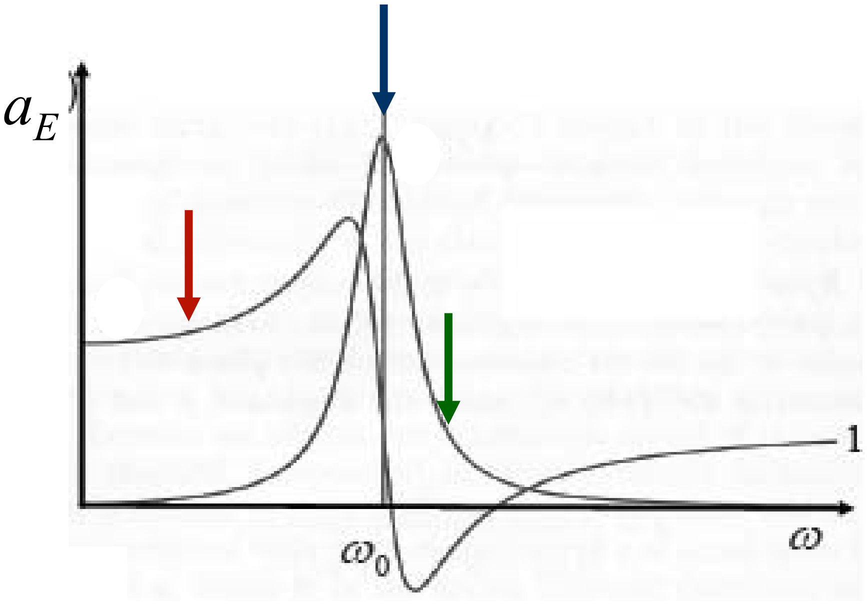
Dipole moment of the resulting dipole

$$\mathbf{p} = \boxed{-\frac{e^2}{m_e}\frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}}\mathbf{E}_L$$

Atomic polarizability, a_E

The forced oscillator

$$\mathbf{p} = \frac{e^2}{m_e}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma}\mathbf{E}_L$$



	Force	Oscillator	
Below resonance $\omega \ll \omega_0$	−	●	Weak vibration. In phase.
On resonance $\omega = \omega_0$	−	●	Strong vibration.9 0° out of phase.
Above resonance $\omega \gg \omega_0$	−	●	Weak vibration. 180° out of phase.

Slide by ….

Slide by R. Trebino

Polarization and susceptibility of a system of atoms

$$\mathbf{p} = -\frac{e^2}{m_e}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma}\mathbf{E}_L$$

For a system of *n* atoms per unit volume, of one electron per atom, one can calculate polarization, as

$$\mathbf{P} = \frac{1}{V}\sum_i \mathbf{p}_i = \varepsilon_0(-\frac{ne^2}{\varepsilon_0 m_e}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma})\mathbf{E}_L$$

For rare systems, local field E$_L$ equals averaged field E

$$\mathbf{P} = \frac{1}{V}\sum_i \mathbf{p}_i = \varepsilon_0\boxed{(-\frac{ne^2}{\varepsilon_0 m_e}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma})}\mathbf{E}$$

χ_e **susceptibility**

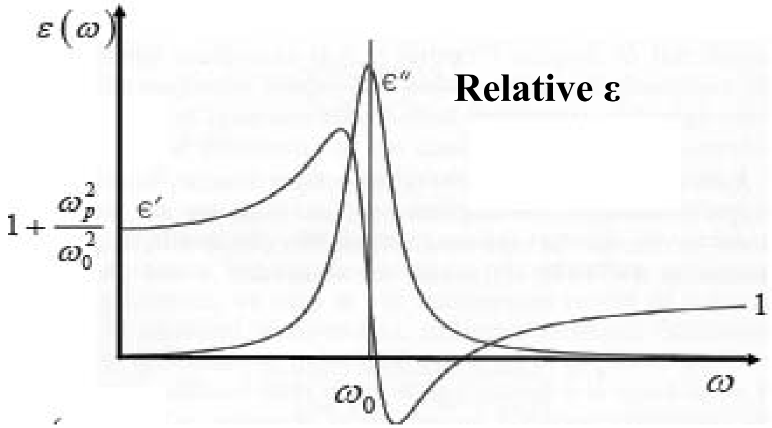
This is the Lorenz model

Dielectric function of a system of atoms

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$

$$\omega_p^2 = \frac{ne^2}{\varepsilon_0 m_e}$$

$$\boxed{\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\gamma}\right)}$$



Lorenz-type ε

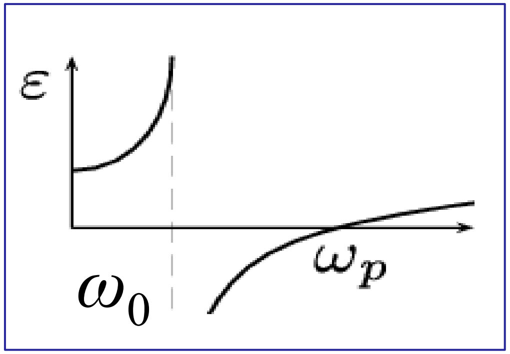
Realistic atoms have many resonance frequencies

What ω_0 and γ represent?

ω_0 : frequency of any atomic or molecular transition, between electronic, oscillating, or rotating levels

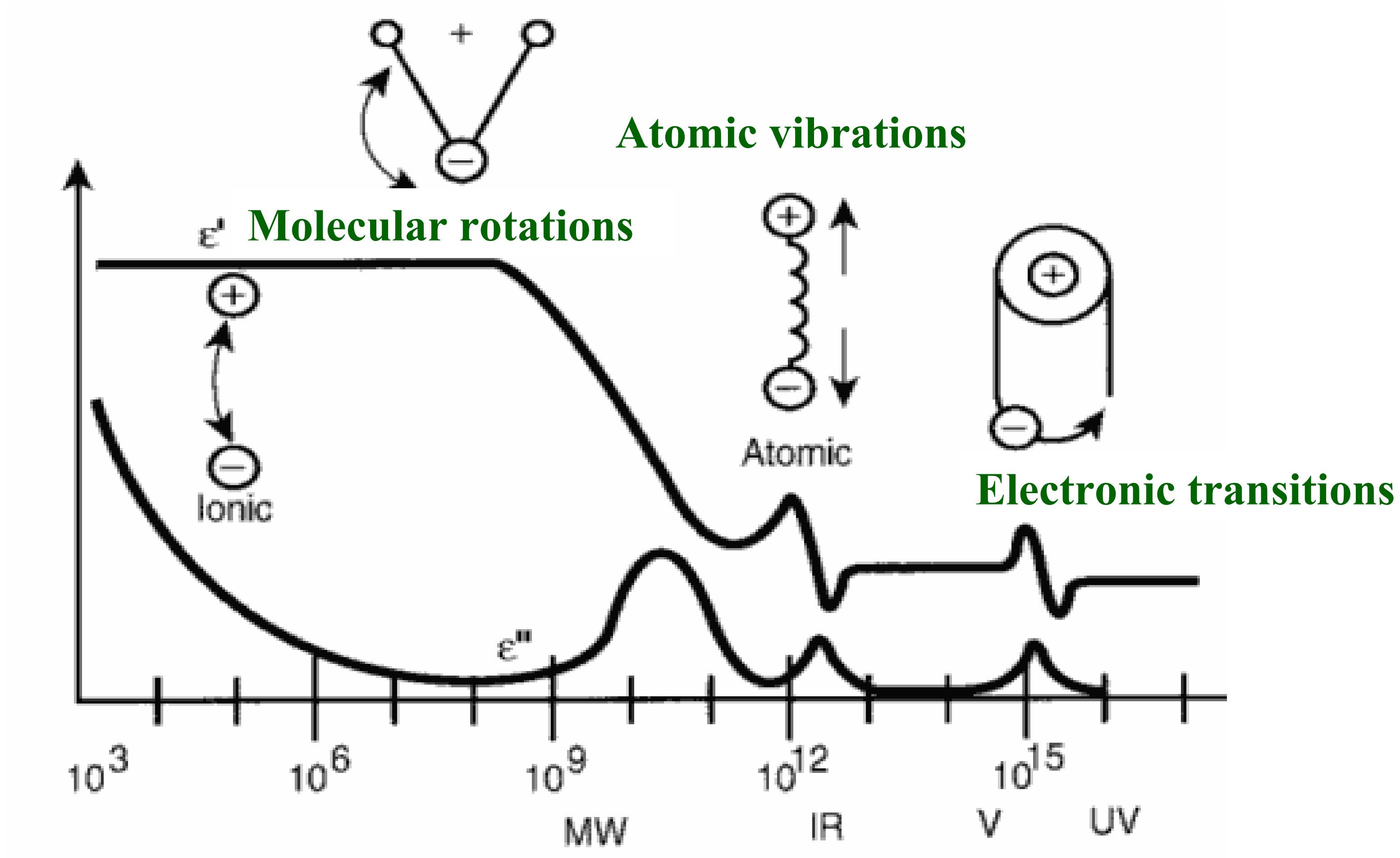
γ : losses due to collisions, spontaneous emission

Re



For γ=0

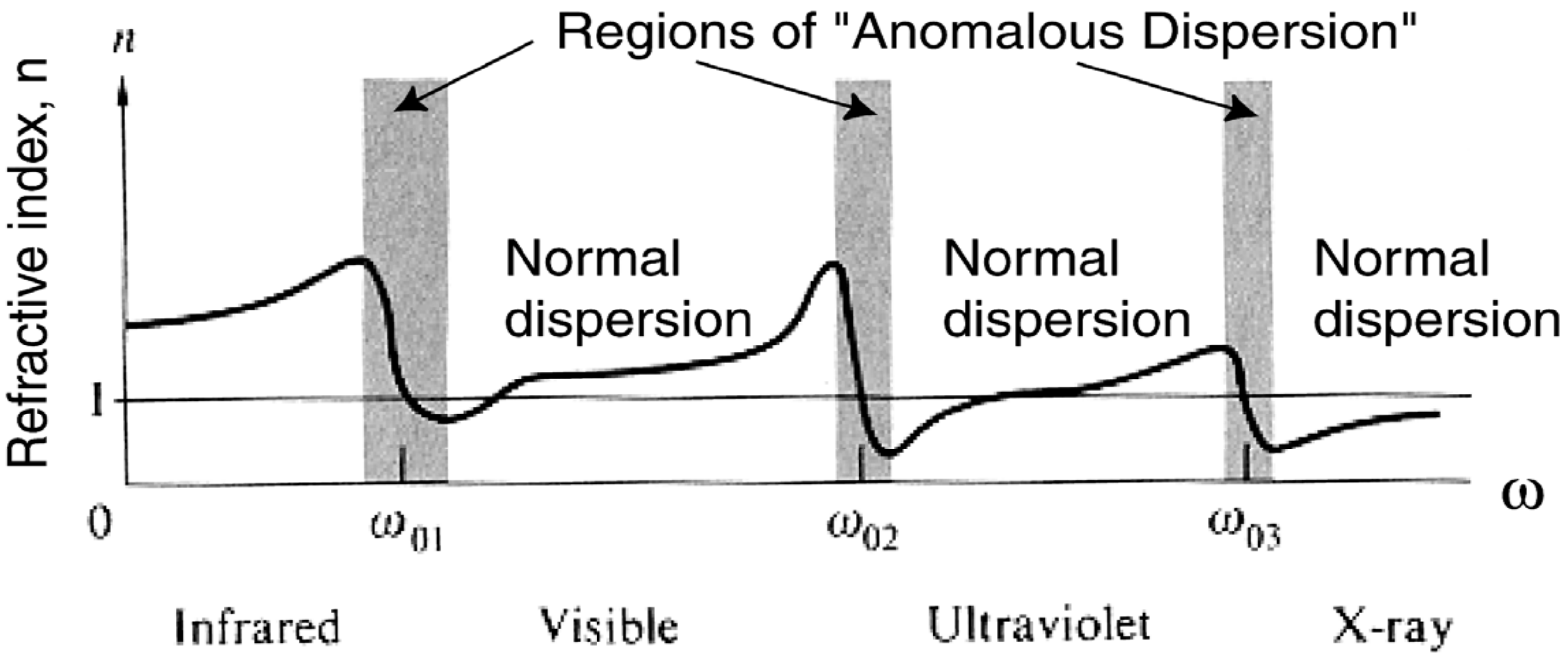
Dielectric function of a system of atoms



For high frequencies ε→1

Picture from Wikipedia

Refractive index of a system of atoms



Picture by R. Trebino

Anomalous dispersion: negative slope

Electronic resonances usually occur in the UV; vibrational and rotational resonances occur in the IR; inner-shell electronic resonances occur in the x-ray region

Electric response of metals?

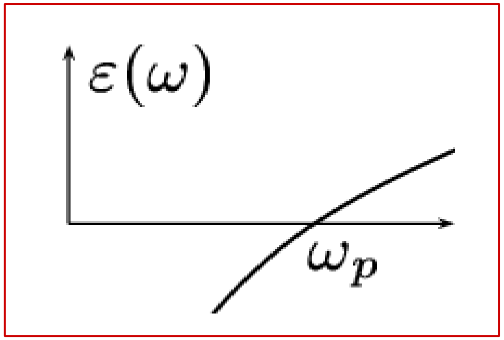
Forces acting on a free electron:

$$m_e\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}_{damping}+\underset{\underset{-m_e\omega_0^2\mathbf{r}}{||}}{\xcancel{\mathbf{F}_{restoring}}}+\mathbf{F}_{external}$$

With same procedure as for bind electrons ➡

$$\boxed{\varepsilon=\varepsilon_0(1-\frac{\omega_p^2}{\omega^2+i\omega\gamma})}$$

Drude-type permittivity



$$\omega_p^2=\frac{ne^2}{\varepsilon_0 m_e}$$

Plasma frequency

Conductivity σ:

$$\varepsilon=\varepsilon_0+i\frac{\sigma}{\omega}\Rightarrow\sigma=\frac{\omega_p^2}{\gamma-i\omega}$$

Constitutive relations – Materials classification

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega)$$
$$\mathbf{B}(\omega) = \mu(\omega)\mathbf{H}(\omega)$$

??

Only for isotropic media

Constitutive relations – Materials classification

The most general relations

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} + \overleftrightarrow{\xi}\mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta}\mathbf{E} + \overleftrightarrow{\mu}\mathbf{H}$$

In matrix form

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overleftrightarrow{\varepsilon} & \overleftrightarrow{\xi} \\ \overleftrightarrow{\zeta} & \overleftrightarrow{\mu} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

$\overleftrightarrow{\varepsilon}, \overleftrightarrow{\mu}, \overleftrightarrow{\xi}, \overleftrightarrow{\zeta}$ 3×3 matrices, i.e. up to 36 independent parameters

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \begin{pmatrix} \xi_{xx} & \xi_{xy} & \xi_{xz} \\ \xi_{yx} & \xi_{yy} & \xi_{yz} \\ \xi_{zx} & \xi_{zy} & \xi_{zz} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

Magnetoelectric coupling

(coupling of electric and magnetic response)

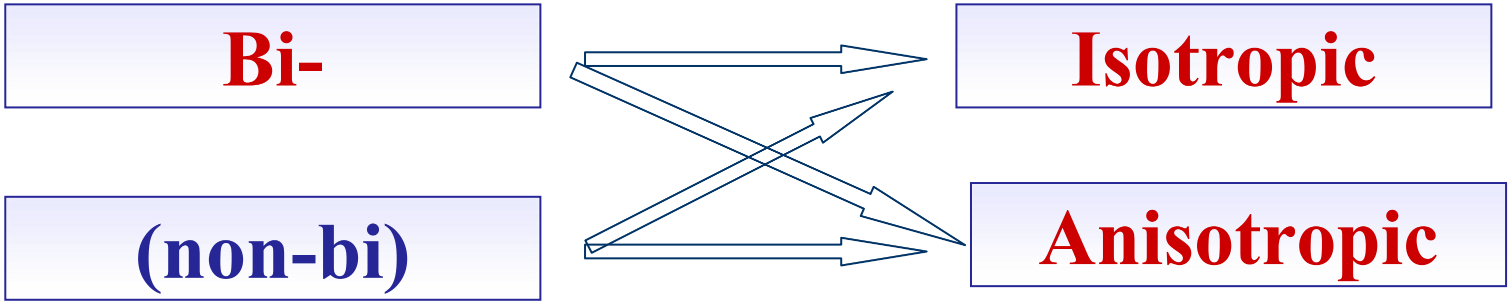
Such a medium is a Bianisotropic medium

Bi = each flux depends on both (two) fields

Anisotropic = at least one of the parameters is tensor

Constitutive relations – Materials classification

In general a medium can be



Bi-

(non-bi)

Isotropic

Anisotropic

Bi-

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} + \overleftrightarrow{\xi}\mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta}\mathbf{E} + \overleftrightarrow{\mu}\mathbf{H}$$

(non-bi)

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E}$$

$$\mathbf{B} = \overleftrightarrow{\mu}\mathbf{H}$$

Isotropic medium

ALL the parameters ε, μ are diagonal tensors of the form

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \quad \textbf{AND} \quad \overleftrightarrow{\mu} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

Thus

$$\begin{aligned} \mathbf{D} &= \overleftrightarrow{\varepsilon}\mathbf{E} \\ \mathbf{B} &= \overleftrightarrow{\mu}\mathbf{H} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \mathbf{D} &= \varepsilon\mathbf{E} \\ \mathbf{B} &= \mu\mathbf{H} \end{aligned}$$

i.e. D and E parallel and of the same ratio independently of the direction of wave propagation – same for B and H

Greek: isos=equal, tropos=way, i.e. isotropic=behave in equal way for all directions

Anisotropic medium

At least one of the parameters ε, μ (and ξ, ζ if non zero) is tensor

E.g. $\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} \Leftrightarrow$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

i.e. D and E not parallel and their relation depends on the direction

Often ε or μ are diagonal, i.e.

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Greek: anisos=unequal, tropos=way

Constitutive relations – Materials classification

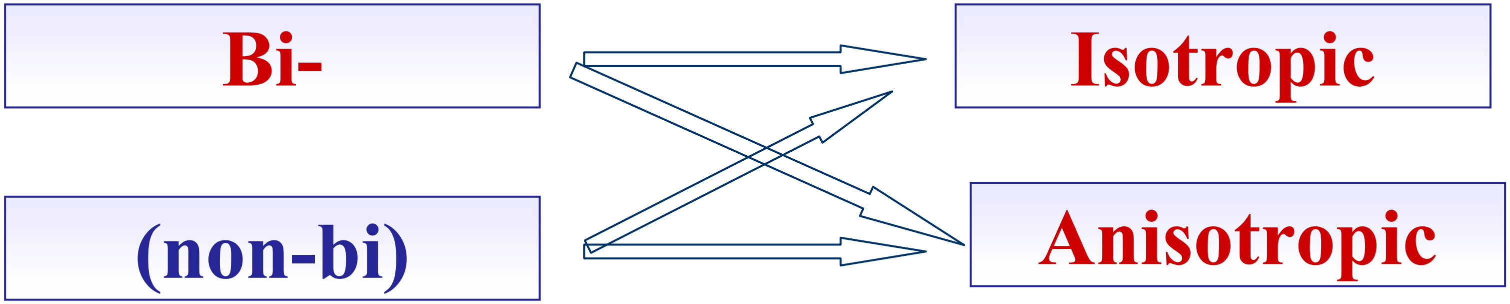
In general a medium can be

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E} + \overleftrightarrow{\xi}\mathbf{H}$$

$$\mathbf{B} = \overleftrightarrow{\zeta}\mathbf{E} + \overleftrightarrow{\mu}\mathbf{H}$$

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\mathbf{E}$$

$$\mathbf{B} = \overleftrightarrow{\mu}\mathbf{H}$$



Independent parameters required for the knowledge of the material:

Bianisotropic: up to 36
Biisotropic: 4
Anisotropic: up to 18
Isotropic: 2

Causality requires*:

$$\overleftrightarrow{\varepsilon}(\omega)^* = \overleftrightarrow{\varepsilon}(-\omega^*), \quad \overleftrightarrow{\mu}(\omega)^* = \overleftrightarrow{\mu}(-\omega^*)$$

Same for $\overleftrightarrow{\xi}, \overleftrightarrow{\zeta}$

***Since e.g.** $\varepsilon(\omega) = \int_0^\infty \varepsilon(t) e^{-i\omega t}\, dt$

Subclasses of anisotropic media: Uniaxial

Uniaxial

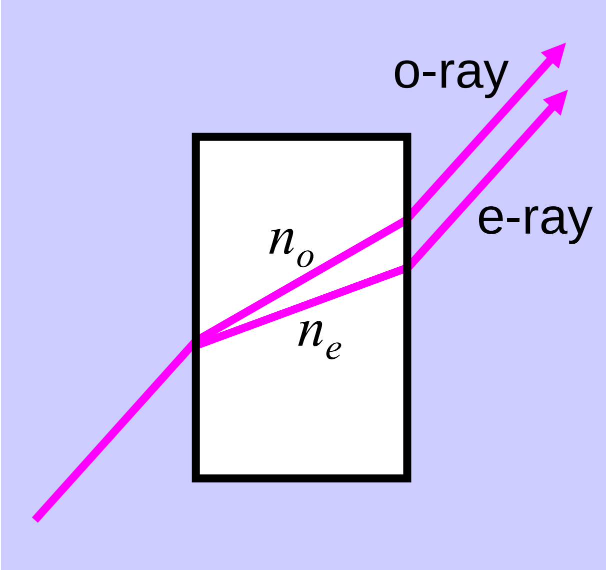
$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

z: optical axis

e.g. some crystals

Birefringence effect:

Two refractive indices: One for light polarized along the optic axis (n_e) and another for light polarized in either of the two directions (n_o)



Picture by R. Trebino

Light polarized along the optic axis is called the extraordinary ray, and light polarized perpendicular to it is called the ordinary ray

Subclasses of anisotropic media

Gyroelectric

$$\overset{\leftrightarrow}{\varepsilon} = \begin{pmatrix} \varepsilon & i\varepsilon_g & 0 \\ -i\varepsilon_g & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

e.g. electron plasma in magnetic field Bz

Gyromagnetic

$$\overset{\leftrightarrow}{\mu} = \begin{pmatrix} \mu & i\mu_g & 0 \\ -i\mu_g & \mu & 0 \\ 0 & 0 & \mu_z \end{pmatrix}$$

e.g. ferrites in magnetic field Bz

Greek: gyro=around

Subclasses of biisotropic media: Chiral

Chiral media (no identical with their mirror images)

$$\mathbf{D} = \varepsilon\mathbf{E} + i\kappa\mathbf{H}$$

$$\mathbf{B} = -i\kappa\mathbf{E} + \mu\mathbf{H}$$

e.g. DNA

Greek: cheri=hand

κ=Chirality parameter

In matrix form:

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \varepsilon & i\kappa \\ -i\kappa & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

Chiral media have different refractive indices for right and left circularly polarized waves

Extreme idealized media

- Perfect electric conductor (PEC)
- Perfect magnetic conductor (PMC)
- Perfect electromagnetic conductor (PEMC)

PEC

$$\varepsilon \to \infty \,, \mu \to 0$$

then

$$\sigma \to \infty, \; z = \sqrt{\frac{\mu}{\varepsilon}} \to 0$$

Small impedance

PMC

$$\varepsilon \to 0 \,, \mu \to \infty$$

then

$$z = \sqrt{\frac{\mu}{\varepsilon}} \to \infty$$

"Infinite" impedance

Main references used for this talk

Books

- J. D. Jackson, *Classical Electrodynamics*, Academic Press
- D. J. Griffiths, *Introduction to Electrodynamics*, John Wiley
- E. N. Economou, *Solid State Physics*, in press
- P. Markos and C. M. Soukoulis, *Wave Propagation: From Electrons to Photonic Crystals and Left-Handed Materials*, Princeton Univ. Press
- J. A. Kong, *Electromagnetic Wave Theory*, EMW Publishing

Internet lectures by

- Martin Savage, Univ. of Washington, savage@phys.washington.edu (http://www.phys.washington.edu/users/savage/Class_122_07/122schedSPR07.html)
- Rick Trebino, Georgia Inst. of Technology, rick.trebino@physics.gatech.edu (http://www.physics.gatech.edu/gcuo/lectures/)
- Stefan Linden & Martin Wegener, Univ.of Karlsruhe, stefan.linden@physik.uni-karlsruhe.de (private commun.)
- Vladimir Shalaev, Purdue Univ., shalaev@purdue.edu (http://www.nanohub.org/courses/nanophotonics/)
- Sergei Tretyakov and Ari Shivola, Helsinki Univ. of Technology, sergei.tretyakov@tkk.fi (http://www.metamorphose-vi.org/)
- Wikipedia

Thank you !!