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Resonant and antiresonant frequency dependence of the effective parameters of metamaterials

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We present a numerical study of the electromagnetic response of the metamaterial elements that are used to construct materials with negative refractive index. For an array of split ring resonators (SRR) we find that the resonant behavior of the effective magnetic permeability is accompanied by an antiresonant behavior of the effective permittivity. In addition, the imaginary parts of the effective permittivity and permeability are opposite in sign. We also observe an identical resonant versus antiresonant frequency dependence of the effective materials parameters for a periodic array of thin metallic wires with cuts placed periodically along the length of the wire, with roles of the permittivity and permeability reversed from the SRR case. We show in a simple manner that the finite unit cell size is responsible for the antiresonant behavior.

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The recent development of metamaterials with negative refractive index or double negative (DNG) metamaterials [1] has confirmed that structures can be fabricated that can be interpreted as having both a negative effective permittivity ϵ and a negative effective permeability μ simultaneously. Since the original microwave experiment of Smith *et al.* [2] various new samples were prepared [3,4], all of which have been shown to exhibit a pass band in which the permittivity and permeability are both negative. These materials have been used to demonstrate negative refraction of electromagnetic waves [5], a phenomenon predicted by Veselago [6]. Subsequent experiments [7] have reaffirmed the property of negative refraction, giving strong support to the interpretation that these metamaterials can be correctly described by negative permittivity and negative permeability [8,9].

There is also an increasing amount of numerical work [10–13] in which the transmission and reflection of electromagnetic wave is calculated for a finite length of metamaterial. For a finite slab of continuous material, the complex transmission and reflection coefficients are directly related to the refractive index n and impedance z associated with the slab, which can in turn be expressed in terms of permittivity ϵ and permeability μ. A retrieval procedure can then be applied to find material parameters for a finite length of metamaterial, with the assumption that the material can be treated as continuous. A retrieval process was applied in Ref. [14], and confirmed that a medium composed of split ring resonators (SRRs) and wires could indeed be characterized by effective ϵ and μ whose real parts were both negative over a finite frequency band, as was the real part of the refractive index n.

The retrieval process, however, uncovers some unexpected effects. For the SRR medium, for instance, the real part of the effective permittivity ϵ' exhibits an antiresonant frequency dependence in the same frequency region where the permeability undergoes its resonance. This antiresonance can be seen in the composite SRR+wire negative index medium as well. The antiresonance in the real part of the permittivity is also accompanied by antiresonant behavior in the imaginary part of the permittivity ϵ'', which exhibits an absorption peak opposite in sign to that of the imaginary part of the permeability. Assuming that waves have a time dependence of $\exp(-i\omega t)$, one would expect the imaginary parts of both the permittivity and permeability to have positive values at all frequencies, since the material is passive [15].

The aim of the present Rapid Communication is to show that the antiresonant behavior of the material parameter is an intrinsic property of a metamaterial, a consequence of the finite spatial periodicity. To illustrate this point, we present the retrieved material parameters for two types of metamaterial media that have been used to form negative refractive index composites: the first medium comprises an array of SRRs that exhibit a resonant permeability and antiresonant permittivity. The second medium comprises an array of cut metallic wires, which exhibits resonant permittivity and antiresonant permeability.

We used the transfer matrix method to simulate numerically the transmission of the electromagnetic waves through the metamaterials. Transmission data for an array of SRR were published elsewhere [10,13] and will not be repeated here. To the best of our knowledge, the analysis of the array of cut wires has not been presented in the literature yet. Figure 1 shows the frequency dependence of the transmission and absorption for an array of cut wires. As expected, the system exhibits a band gap for frequencies $f_0<f<f_p$. Here, f_0 is the resonance frequency and f_p is the plasma frequency. The transmission data are similar to that for an array of the SRR. However, contrary to an array of SRR, the system exhibits the resonant behavior of the effective permittivity at $f=f_0$.

From the transmission and reflection data we calculate the effective permittivity and permeability. Details of the method were published elsewhere [14]. The method is based on the assumption that the system is homogeneous. Textbook formulas for the transmission and reflection of the slab of width

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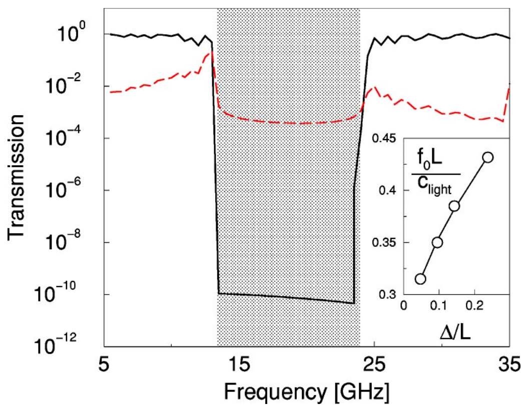

FIG. 1. Transmission of the EM wave through the periodic lattice of thin metallic wires. Wires are parallel with the y axis, and the EM wave, polarized with $\vec{E}\|y$, propagates in the z direction. The system is infinite in the x and z direction, and 60 rows of wires are considered along the propagation direction. The structure is characterized by four length parameters: the wire thickness w, gap Δ, the wire length L, and the lattice period (mutual distance of wires) a. In the present simulations, the lattice constant is $a=3.66$ mm, the thickness of the wire is $w=0.33$ mm, the length of the wire is $L=7$ mm, and the cut of the wire is $\Delta=0.33$ mm. Data show the drop of the transmission (solid line) at frequency $f_0\approx 13.35$ GHz. Dashed line is absorption. We found that the resonant frequency f_0 is almost independent of the lattice constant a ($1.66\leqslant a\leqslant 5$ mm). However, it depends strongly on the gap Δ. Inset shows how the frequency f_0 depends on the ratio Δ/L. Solid line is the fit $f_0=(c_{\rm light}/\sqrt{2\pi}L)[a_0\ln(L/\Delta)+a_1]^{-1/2}$ with $a_0=0.48$ and $a_1=0.17$. Plasma frequency f_p depends on the lattice period a. For $a=3.66$ mm we found numerically $f_p\approx 24.5$ GHz which agrees with predictions of Sarychev and Shalaev [21], indicating that the value of the plasma frequency is not influenced by the wire cut.

d are then inverted to obtain effective impedance z and effective refractive index n. Permittivity and permeability are obtained from relations

$$n=\sqrt{\epsilon\mu} \quad \text{and} \quad z=\sqrt{\mu/\epsilon}. \tag{1}$$

Typical frequency dependence of the effective parameters of an array of SRR and array of cut wires are shown in Figs. 2 and 3, respectively. Both structures exhibit qualitatively the same behavior: there is a resonant frequency interval, in which one effective parameter is negative (μ' for SRR, ϵ' for cut wires). The resonant behavior of this parameter at the left border of the band gap is clearly visible. The frequency dependence of the second parameter is antiresonant. Its real part decreases to zero at $f=f_0$ and imaginary part is negative for $f>f_0$. Notice the qualitative similarity of both systems: data are qualitatively the same, and differ only in the exchange of ϵ and μ (which is equivalent to the transformation $z\to 1/z$).

The antiresonant behavior of the effective parameter has its origin in the finite lattice period a associated with the metamaterial structure. One manifestation of the lattice periodicity is that there is a maximal wave number, given by $k_{\max}=\pi/a$ [20]. If we assume that the metamaterial can be treated as a continuous medium with an index of refraction n,

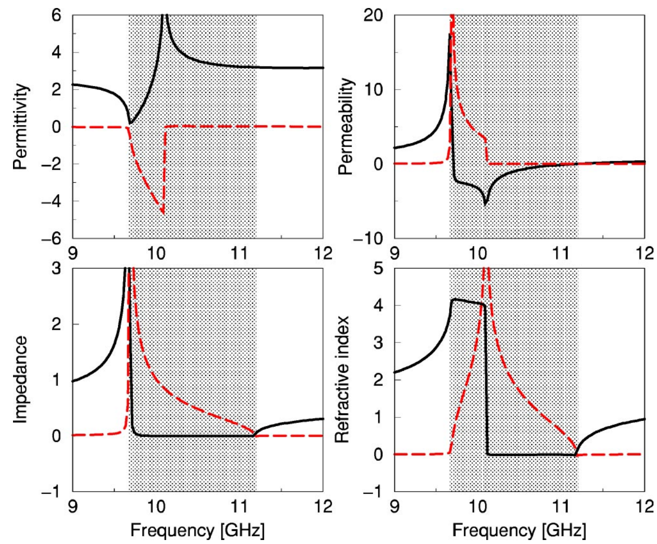

FIG. 2. Effective parameters for an array of split ring resonators (solid lines—real part, dashed lines—imaginary part). The resonant behavior of the effective permeability μ at frequency $f_0=9.66$ GHz is clearly visible. Shaded area shows the resonance frequency interval in which μ' is negative. Note the antiresonant behavior of ϵ. Note also that ϵ'' is negative. The sharp discontinuity in ϵ'' is due to the extremely fast decrease of n' to zero in the right part of the resonance interval. The size of SRR is 3 mm, the size of unit cell is $L_x\times L_y\times L_z=3.33\times 3.66\times 3.66$ mm. The SRR lies in the $x=0$ plane, EM wave propagates along the z direction and is $E\|y$ polarized.

then the definition of $n=c_{\rm light}k/\omega$ shows that n is necessarily bounded. The generic $\omega(k)$ dispersion diagram of a resonant periodic structure results from the coupling of a dispersionless resonance curve at the resonant frequency $\omega_0=2\pi f_0$ and the light line $f=c_{\rm light}/k$ (see Ref. [2] for details). The

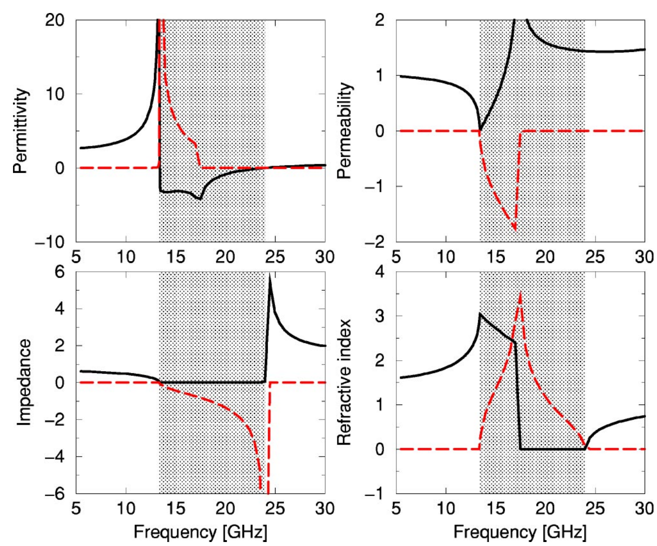

FIG. 3. Effective parameters of the periodic lattice of cut wires. Resonant behavior of the effective permittivity ϵ as well as antiresonant behavior of effective permeability μ are clearly visible. The real part of permeability μ' is zero at f_0, and the imaginary part of the permeability μ'' is negative for $f>f_0$.

result is a lower branch that extends from zero frequency to the resonance frequency f_0, followed by a band gap that extends from f_0 to f_p, followed by an upper branch that extends upwards in the frequency from f_p.

Because the resonant frequency of a typical metamaterial element implies a free space wavelength much longer than the unit cell size, an effective medium approach has been applied that results in a characterization of metamaterial in terms of bulk ϵ and μ. The square root of the product $\epsilon\mu$ is the refractive index, which must be consistent with that determined from the dispersion diagram. To understand the generic properties associated with a resonant system in a periodic structure, we start by considering the dispersion characteristics of the system near the resonant frequency, where we have

$$\omega \sim \omega_0 - \frac{1}{\alpha^2}\left(k - \frac{\pi}{a}\right)^2, \tag{2}$$

where α is a real number. Solving Eq. (2) for k and using $n = c_{\text{light}} k/\omega$, we find an approximate expression for the refractive index

$$n(f) \approx \frac{c_{\text{light}}}{2\pi f_0}\left(\frac{\pi}{a} - \alpha\sqrt{\omega_0 - \omega}\right). \tag{3}$$

The maximum value of the refractive index at the resonance frequency is thus

$$n_{\max} = \frac{c_{\text{light}}}{\omega_0}\frac{\pi}{a} \tag{4}$$

determined only by the resonant frequency of the element and the periodicity.

For the composite structures analyzed in this paper, Eq. (4) gives $n_{\max} = 4.24$ for an array of SRRs and $n_{\max} = 3$ for the cut wires array. When comparing $n_{\max}$ with data presented in Figs. 2 and 3, we see that n' indeed does not exceed these limits.

Assume that the effective parameter x (x represents the effective permittivity for the cut wires system and the effective permeability for the SRR system) exhibits a resonant form corresponding to

$$x(f) = 1 - \frac{F f^2}{f^2 - f_0^2 + i\gamma f}, \tag{5}$$

where γ is a damping factor and F is a filling factor (fraction of volume of the metallic components). At resonance, the imaginary part $x''(\omega)$ becomes

$$x''(\omega) = (1 - F)\frac{\omega_p^2 - \omega_0^2}{\gamma\omega_0} > 0 \tag{6}$$

$[\omega_p^2 = \omega_0^2/(1-F)]$ and is greater than zero, as expected.

If we require that the index n calculated from the dispersion curves be consistent with the bulk permittivity and permeability parameters, then we must have $n(\omega) = \sqrt{x(\omega)y(\omega)}$. Near the resonance frequency this implies that the second effective parameters $y(\omega)$ behave as

$$y(\omega) = \frac{n^2(\omega)}{x(\omega)}. \tag{7}$$

Comparing the expression for $y(\omega)$ with that for $x(\omega)$ we see that the poles and zeros of x and y are reversed, as long as n is bounded with the form given by Eq. (4). Moreover, for small $n''(\omega)$, we immediately see that the product of imaginary parts $x''(\omega)y''(\omega)$ is negative. To be more specific, with the help of Eq. (1) we obtain that

$$\epsilon''\mu'' = \frac{1}{|z|^2}[(n''z')^2 - (n'z'')^2]. \tag{8}$$

One sees that the sign of $\epsilon''\mu''$ is fully determined by the right-hand side of Eq. (8). We are not aware about any physical requirement which prevents the rhs of Eq. (8) to be negative. In fact, data presented in Figs. 2 and 3 show clearly that

$$|n''z'| \leq |n'z''| \tag{9}$$

in the left part of the resonance gap, since $|z'| < |z''|$ and $|n''| < |n'|$. Thus, in agreement with Eq. (7) we conclude that the opposite sign of ϵ'' and μ'' is a consequence of small transmission losses in the structure. The same conclusion was derived in Ref. [13] for the DNG metamaterials.

The fact that imaginary part of either permittivity or permeability is negative seems to contradict our physical intuition. There are indeed no doubts that imaginary part of the response function ξ'' (ξ is either permittivity or permeability in our case) must be positive when only one external force (electric or magnetic field) acts on the body [16,17] since the energy W dissipated inside the medium is proportional to ξ''. In the present case, however, we analyze simultaneous response of both electric and magnetic field to the metamaterial, and the dissipated energy is given as a sum

$$W = \frac{1}{4\pi}\int d\omega\,\omega[\epsilon''(\omega)|E(\omega)|^2 + \mu''(\omega)|H(\omega)|^2]. \tag{10}$$

The condition $W > 0$ does not require that ϵ'' and μ'' must be simultaneously positive [18]. Negative imaginary part of the permeability of metamaterial was numerically obtained also in Ref. [19].

In conclusion, we presented the numerical analysis of the effective parameters of two metamaterials: an array of SRR and an array of thin metallic cut wires. We show that the effective parameters of these systems exhibit resonant and antiresonant behavior similar to that found recently in the double-negative metamaterials. We suggest that the antiresonant behavior is caused by the requirement that the refractive index must be bounded in the structures which possess finite spatial periodicity. As the spatial periodicity is an unavoidable property of the metamaterials, we conclude that the observed seemingly unphysical behavior of effective material parameters is an intrinsic property of composites, which cannot be avoided, for instance, by decreasing the size of the unit cell.

The electromagnetic response of metamaterials is usually embodied in a description involving bulk, continuous, frequency dependent permittivity and permeability tensors. This description, however, is only approximate, as spatial dispersion is always present to some degree in metamaterials. Thus, our retrieval procedure, which uses the formulas for transmission and reflection of the homogeneous slab, might not be applicable in the neighborhood of the resonance frequency, where the wavelength of the electromagnetic wave inside the composite is already comparable with the spatial period.

Besides the analysis of the effective parameters, our results are interesting also for further development of new double-negative metamaterials, in which the lattice of cut wires might find new interesting applications [13,22].

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