Metal-insulator transitions in anisotropic two-dimensional systems

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Several phenomena related to the critical behavior of noninteracting electrons in a disordered two-dimensional tight-binding system with a magnetic field are studied. Localization lengths, critical exponents, and density of states are computed using transfer-matrix techniques. Scaling functions of isotropic systems are recovered once the dimension of the system in each direction is chosen proportional to the localization length. It is also found that the critical point is independent of the propagation direction, and that the critical exponents for the localization length for both propagating directions are equal to that of the isotropic system, $\nu \approx \frac{7}{3}$. We also calculate the critical value Λ_c of the scaling function for both the isotropic and the anisotropic system. It is found that $\Lambda_c^{\text{iso}} = \sqrt{\Lambda_c^x \Lambda_c^y}$. Detailed numerical studies of the density of states $n(E)$ for the isotropic system reveal that for an appreciable amount of disorder, the critical energy is off the band center.

I. INTRODUCTION

The problem of Anderson localization[1] in anisotropic systems has attracted considerable attention[2–5] recently. It is generally accepted[2] that anisotropy does not change the universality class and that the isotropic results are recovered once a proper scaling of the anisotropic results is performed. If the dimension of the system size is chosen to be directly proportional to the localization length, the system should be effectively isotropic. The difficulty in implementing such a procedure lies in the fact that the localization lengths are usually not known *a priori*. It was found through detailed numerical calculations[2] that this scaling indeed works. It was also shown[6] that the probability distributions of the conductance in the two directions are exactly equal to each other, provided that the ratio of the sides of the rectangle is proportional to the ratio of the localization lengths in the two directions. These scaling results were obtained for an anisotropic system where all the states were localized.

It is well known[1] that noninteracting electrons are localized in two-dimensional (2D) disordered systems. There are, however, some exceptions to this rule. These include electrons having strong spin-orbit coupling,[7] integer quantum Hall systems,[8] and tight-binding models with random magnetic fields.[9] The best known example is the integer quantum Hall plateau transition occurring in a 2D noninteracting system in a strong magnetic field. Extended states do not exist as a result of Anderson localization except at a singular energy near the center of each of the Landau subbands.[8,10] The localization length diverges at these critical energies E_c as $\xi \propto |E - E_c|^{-\nu}$.

Another important point is the universality of the conductance at the critical point of the Anderson transition for the anisotropic system.[4] From the generalized scaling functions, it has been established that the geometric mean of the critical value Λ_c of the scaling function $\Lambda \stackrel{\text{def}}{=} (\lambda_M / M)$ (as a function of ξ/M) is a constant independent of the strength of the anisotropy. (Here λ_M denotes the finite-size localization length of a quasi-one-dimensional strip of finite width M.) Numerical calculations in both two-[2] and three-dimensional[4] disordered anisotropic systems support this claim. However,

the same is not true for the conductance. Numerical calculations[4] in three-dimensional anisotropic systems do not support a universal value of the conductance for the geometric mean. This might be due to too small sizes used in the 3D system or to a lack of universality of the conductance at the critical point.

In this paper, we investigate the scaling properties of the finite-size localization length λ_M and the critical value Λ_c of the scaling function in a two-dimensional system described by a tight-binding model in the presence of a magnetic field. Both the isotropic case as well as the anisotropic case will be examined. This is perhaps the simplest system that exhibits the correct behavior of the metal-to-insulator transition. To our knowledge, no such calculations have been previously reported for the anisotropic tight-binding model with a constant magnetic field. Some of the questions we try to answer are as follows: How does the anisotropy affect the critical behavior, especially, will there be one or two critical exponents for the localization lengths? How do the anisotropic quantities relate to the corresponding isotropic ones, especially, can we expect the geometric mean of the two anisotropic values to equal the isotropic value? What are the values for the scaling functions at the critical point? In Sec. II, we describe the model and the numerical methods we used. In Sec. III, we present and discuss our numerical results. In Sec. IV, we summarize the conclusions of this work.

II. MODEL AND METHODS

In the tight-binding model, one has the Hamiltonian

$$\mathcal{H} = \sum_i |i\rangle \varepsilon_i \langle i| + \sum_{i,j}{}' |i\rangle V_{ij} \langle j|, \tag{1}$$

where the summations run over lattice sites i and j. We consider only nearest-neighbor interaction in the hopping integrals V_{ij}. The effects of an external magnetic field, characterized by a vector potential $\mathbf{A}$ ($\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$), enter the model via phases of the hopping integrals with

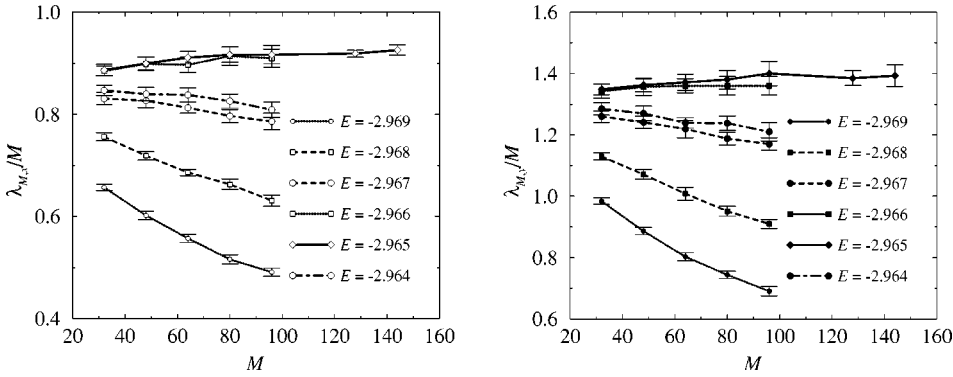$$V_{ij} = t_{ij} e^{-2\pi i (e/h) \int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A}(\mathbf{r}) d\mathbf{r}}, \tag{2}$$

FIG. 1. The critical values of the scaling functions can be obtained from the large-M limit of λ_M/M at the critical energy. From the almost symmetrical behavior of the values for the noncritical energies at either side of the critical one, we assume that the value for E_c is between -2.966 and -2.965, but closer to the second one. Left: difficult-hopping direction; right: easy-hopping direction.

the integral connecting lattice sites i and j by a straight line. In two dimensions with a magnetic induction **B** perpendicular to the plane of the system, one can choose the gauge for the vector potential in such a manner that the phases vanish in one direction within the plane and are integer multiples of some number $2\pi\alpha$ in the other direction, such that the value of α completely characterizes the influences of the magnetic field on the system. In particular, the denominator of a rational α equals the number of bands in the density of states of the system without disorder. Introducing anisotropy into the system by choosing different amplitudes t_{ij} in the two directions within the plane will effect only the position of these bands, not their number. We bring disorder into the system by independently choosing all the site energies ε_i from a rectangular distribution of width W centered at 0; thus W is a measure of disorder strength. Both W and E are measured in units of the largest hopping matrix element t, which is taken to be unity.

As our main method, we use the transfer-matrix method,[1] where a matrix T_M connects the amplitudes of a state at both ends of a quasi-one-dimensional strip of width M and length $N\gg M$. Due to the anisotropy, we have to do this in the two spatial directions separately. Therefore, we get two sets of parameters $\lambda_{M,x}$ and $\lambda_{M,y}$, which lead to two separate localization lengths ξ_x and ξ_y in the x and y direction, respectively. Scaling of the data is used to improve on the values of ξ_x and ξ_y, which are then analyzed to find the critical energies, where the localization lengths diverge as well as the critical exponents of these divergences.

We obtain the density of states by using a Lanczos procedure[11] to diagonalize the Hamiltonian on squares of (linear) size L. The energy-level separation distribution function $p(s)$ should deviate markedly from a Poisson distribution for the local density of states around the critical energy, approaching the Wigner distribution for the unitary ensemble.[12,13] We also use the density of states to show that for sufficiently strong disorder the critical energy does not necessarily coincide with the band center.

III. RESULTS

To obtain the critical energy E_c for the anisotropic tight-binding model, first we calculate $\lambda_{M,x}$ and $\lambda_{M,y}$ for different strip widths M and energies E above and below the critical energy E_c. As the exact position of E_c varies with the dis-

order strength W, the hopping integral $t_x < 1$ in the difficult hopping direction, as well as the magnetic field parameter α, we restrict our investigation to one set of these parameters $W = 0.1$, $t_x = 0.8$, and $\alpha = \frac{1}{8}$. The data for the more localized states show that M/λ_M versus M is a straight line. The inverse slope of each of these lines gives a first estimate for the localization lengths ξ_x or ξ_y, respectively, thus the smaller the slope, the more extended are the corresponding eigenstates of the system. For energies closer to E_c, the lines would be essentially horizontal. In order to accurately obtain E_c, we have systematically calculated $\lambda_{M,x}$ and $\lambda_{M,y}$ for large M. The results are shown in Fig. 1, where we plot $\lambda_{M,x}$ and $\lambda_{M,y}$ for the anisotropic case for energies very close to E_c. From Fig. 1, we can confirm the existence of an extended state. Notice that λ_M/M decreases as a function of M, which signifies localized states. For localized states, λ_M eventually reaches its large-M limit, which is a constant, and therefore λ_M/M decreases as M increases. However, as can be seen from Fig. 1, at the critical energy E_c, λ_M/M saturates to a constant due to the absence of length scales. For the case studied ($W = 0.1$, $t_y = 1.0$, $t_x = 0.8$, and $\alpha = \frac{1}{8}$), we find that the critical energy E_c is between -2.966 and -2.965, but closer to the second value. From Fig. 1, we can also obtain the critical values Λ_c of λ_M/M for both directions of propagation. We find $\Lambda_c^x = 0.92 \pm 0.01$ and $\Lambda_c^y = 1.39 \pm 0.01$ with a geometric mean of 1.13 ± 0.01.

To confirm that the geometric mean Λ_c of the two anisotropic values is related to the value for the isotropic case, we have calculated systematically λ_M/M versus M for the isotropic system ($W = 4.0$, $t_x = t_y = 1.0$, and $\alpha = \frac{1}{8}$) for very large values of M. These results are shown in Fig. 2. From

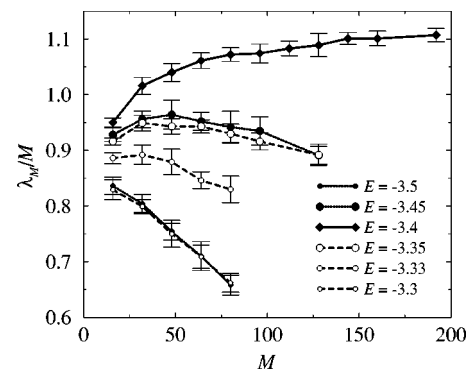

FIG. 2. The critical value of the isotropic scaling function. From the large-M data, we estimate it to be 1.10 ± 0.03.
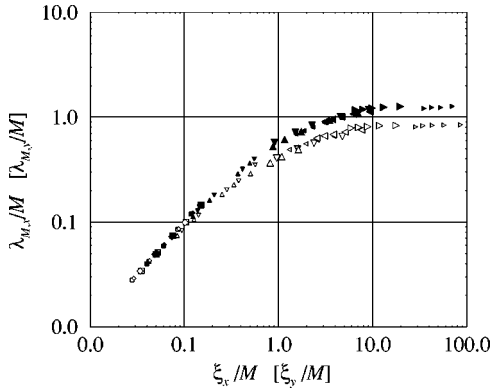
FIG. 3. The scaling functions for the difficult- (open symbols) and easy- (filled symbols) hopping directions λ_M/M as a function of ξ/M. The localization lengths have been adjusted to better fit the data to a smooth curve. Energies are -3.0, -2.99, -2.98, -2.97, -2.969, -2.968, -2.967, -2.964, -2.96, -2.95, -2.94, and -2.93.

Fig. 2, we obtain that indeed $E_c = -3.40$ in this case, in agreement with previous results[10,14] that used different techniques to get E_c. In addition, Fig. 2 shows clearly that at the critical point of the isotropic system $\Lambda_c^{iso} = 1.10 \pm 0.03$, which is approximately equal to the geometric mean of the two anisotropic values Λ_c^x and Λ_c^y.

The critical value of λ_M/M is related to the exponent α_0 that can be obtained from the multifractal analysis[8] of the eigenfunctions at the critical energy by $\Lambda_c^{-1} = \pi(\alpha_0 - d)$, where d is the Euclidian dimension of the system. Huckestein[26] calculated $\Lambda_c = 1.14 \pm 0.02$ for a real-space model, while Lee *et al.*[27] determine $\Lambda_c = \ln^{-1}(1 + \sqrt{2})$ ≈ 1.13 for a network model, both of which are close to the value $\Lambda_c^{iso} = 1.10 \pm 0.03$ obtained for the isotropic case of the 2D tight-binding model with a constant magnetic field.

The next step is to use the values for the localization lengths obtained in this manner to plot $\lambda_{M,x}/M$ as a function of ξ_x/M and $\lambda_{M,y}/M$ as a function of ξ_y/M. After combining the data for all energies into one graph, one usually has to adjust the values for the localization lengths slightly to make the data fall on a smooth curve. Figure 3 shows that these two functions are independent of the value of E, as expected for one-parameter scaling. However, the two scaling functions differ in their large-ξ limit: the value is higher for the easy-hopping direction. To compensate for this anisotropy effect, we use the following straightforward idea:[2] $\lambda_{M,x}$ ($\lambda_{M,y}$) is a length in the $x(y)$ direction along the length of the strip, so the appropriate scale should be $\xi_x(\xi_y)$. However, M is a length measuring the width of the strip and therefore has to be scaled with the other localization length. Thus we plot $(\lambda_{M,x}/\xi_x)(\xi_y/M)$ versus (ξ_y/M) and $(\lambda_{M,y}/\xi_y)(\xi_x/M)$ versus ξ_x/M in Fig. 4. Not only do we obtain the same scaling function for both, but it is also the same as the isotropic one, which we included for reference. The isotropic case was for $W = 4$ and $\alpha = \frac{1}{8}$. Of course, under the assumption of one-parameter scaling, the form of the (isotropic) scaling function should not depend on the values of W and α directly (as long as neither vanishes completely)
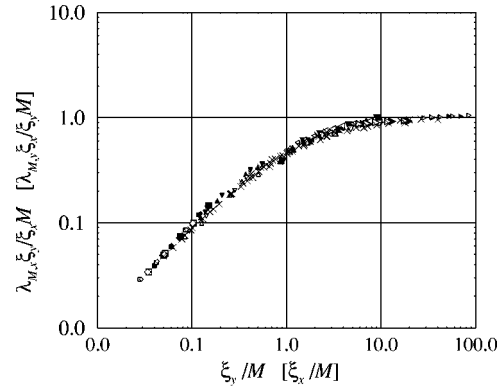

FIG. 4. Plotting the rescaled scaling functions (cf. text) for the difficult- (open symbols) and easy- (filled symbols) hopping directions together with the scaling function for an isotropic system (crosses).

but only parametrically via the localization length $\xi(E, W, \alpha)$. Thus, the product of the two rescaled anisotropic functions equals the square of the isotropic scaling function. Immediately it is seen from this that the isotropic scaling function equals the geometric mean of the two anisotropic scaling functions,

$$\left(\frac{\lambda_M}{M}\right)_{iso} = \sqrt{\frac{\lambda_{M,x}}{M} \frac{\lambda_{M,y}}{M}} \tag{3}$$

as the rescaling factors ξ_x and ξ_y cancel each other. As we have shown before in Fig. 1 and Fig. 2, indeed Eq. (3) is obeyed.

The procedure of fitting the data to a smooth scaling function provides us with more accurate estimates of the localization lengths, which we can now use to determine the critical behavior of ξ. In Fig. 5, we plot the localization lengths
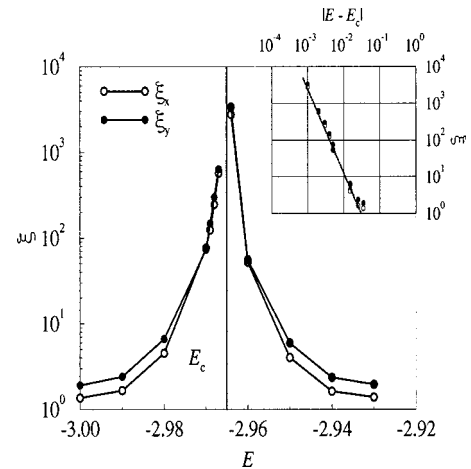

FIG. 5. The divergence of the localization lengths at the critical energy: open symbols, difficult-hopping direction; filled symbols, easy-hopping direction. The values are taken after the adjustments made to obtain Fig. 5. Inset: To extract the critical exponent of the localization lengths we plot ξ vs $|E - E_c|$ in a log-log plot. Both exponents are found to be 2.3 ± 0.1, roughly equal to the theoretical value for the isotropic system.
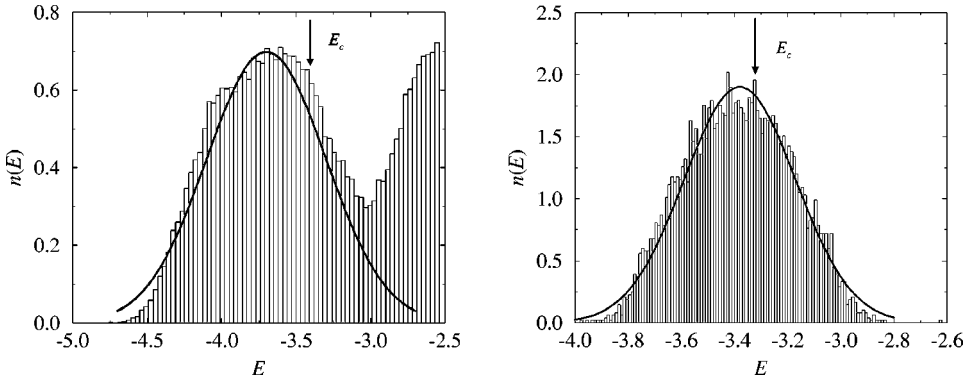
FIG. 6. The density of states for the lowest subband for a disorder strength of $W=4.0$ (left) and $W=2.0$ (right), indicating that the critical energy is off the band center. A fit to a Gaussian distribution suggests that the band center in the $W=2.0$ case is at $E_0 \approx -3.38$, whereas the critical energy is $E_c \approx -3.32$.

as a function of energy. One can clearly see that the states are less localized in the easy-hopping direction, as was to be expected. Figure 5 also allows us to estimate E_c, the energy where the localization length diverges. We expect this critical energy to be independent of the strip orientation, as a higher-dimensional system would undergo a phase transition at this point, and our data give a strong indication that E_c is indeed the same for both directions. We estimate $E_c \approx -2.965 \pm 0.001$. This is consistent with the results obtained in Fig. 1.

The divergence of the localization length near the critical energy is expected to follow a power law,

$$\xi(E) = \xi_0 |E - E_c|^{-\nu}, \qquad (4)$$

with some critical exponent ν. To test this hypothesis, we plot the logarithm of ξ versus the logarithm of $|E-E_c|$. The result is shown in the inset of Fig. 5. That our data follow a straight line rather reasonably reconfirms our estimate for E_c, as the plot obviously is quite sensitive to the choice for that value. Furthermore, both sets of data can be fitted by the same straight line, giving the same critical exponent $\nu \approx 2.3 \pm 0.1$. Once again, this is the same as the isotropic value and very close to the theoretically predicted value[8] of $\frac{7}{3}$ for the isotropic system.

The distribution of energy-level separations in a given energy interval depends on the typical extension of the eigenstates of the system with eigenvalues in that energy region. Spatial overlap of eigenfunctions close in energy helps to delocalize the particle. In a finite system, more of the strongly localized eigenfunctions can be accommodated without significant overlap. The more extended the eigenfunctions become, the more difficult it becomes to fit several into the finite space, and they must be separated in energy. This leads to the phenomenon of level repulsion, known from chaos theory. The corresponding distribution of level separations $s_i = E_i - E_{i-1}$ goes to zero for small s. In contrast, the distribution for a range of localized eigenstates has a maximum at vanishing level separation. More specifically, random matrix theory predicts[13] a Poisson distribution for the localized case and a Wigner distribution for the extended case. We have calculated the distribution of energy-level separations $p(s)$ for the anisotropic system studied in Fig. 1. We find that for an energy range close to $E_c = -2.965$, $p(s)$ is Wigner-like, whereas for the other energy ranges it is Poisson-like.

Level statistics for the isotropic system have been extensively studied by Potempa et al.[14,15] and Batsch et al.,[16,17] proving the validity of the approach in distinguishing localized from extended states. In addition, the level number variance $\Sigma^2(\langle N \rangle) = \chi \langle N \rangle$ has been numerically obtained for the isotropic system,[18] using the Chalker-Coddington network model,[19] and compared with analytical theories,[20] which give for the spectral compressibility $\chi = [d - D(2)]/2d$, where $D(2)$ is the multifractal exponent of the wave function at the critical point.[21] Klesse and Metzler obtain $\chi = 0.124 \pm 0.006$.[18] Numerically obtained values for $D(2)$ include 1.43 ± 0.03 for a continuum model,[22] 1.56 for a network model,[23] and 1.62 ± 0.02 and 1.71 for a tight-binding model.[24,25] Due to the limited size of our systems, we were not able to produce results for χ for our anisotropic model. The number of energy eigenvalues sufficiently close to the critical point is not large enough to give good statistics for the number variance $\Sigma^2(\langle N \rangle)$. This point has to be addressed in the future.

Finally, Fig. 6 shows the positions of E_c for isotropic systems at $W=2$ and $W=4$ to be different from the band center. Although a Gaussian is not the correct form for the density of states, it is usually a reasonable fit. For the stronger disorder, $W=4$, the best approximation is achieved with a Gaussian centered at $E=-3.7$ with a standard deviation of $\sigma = 0.4$. Figure 2 strongly indicates $E_c = -3.4$ (arrow in left panel of Fig. 6). Similarly, for the lesser disorder, $W=2$, a Gaussian centered at $E=-3.38$ with a standard deviation of $\sigma = 0.21$. A plot similar to that for the more strongly disordered case shows that $E_c = -3.32$. However, for $W \leq 1$ the critical energy E_c lies at the center of the Landau band.

IV. CONCLUSIONS

In summary, we have performed a detailed numerical study of the scaling properties of highly anisotropic systems in 2D, with a metal-to-insulator transition. Scaling functions of the isotropic systems are recovered once the dimension of the anisotropic system is chosen to be proportional to the localization length. It is also found that the critical point is independent of the propagation direction and that the critical exponents for the localization length in both propagating di-

rections are equal to that of the isotropic system. The critical value Λ_c of the scaling function for both the isotropic and the anisotropic cases has been calculated. It is obtained that $\Lambda_c^{\text{iso}} = \sqrt{\Lambda_c^x \Lambda_c^y} = 1.10 \pm 0.03$. Finally, density of states calculations revealed that the critical energy lies away from the center of the Landau band.

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