Few- and many-body physics of interaction-bound atoms in optical lattices

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Greece

Outline

- Bose-Hubbard model
  - *Single particle in a lattice*
  - *Two-particle states in a lattice:* **dimers**
  - *Three-particle states in a lattice:* **trimers**

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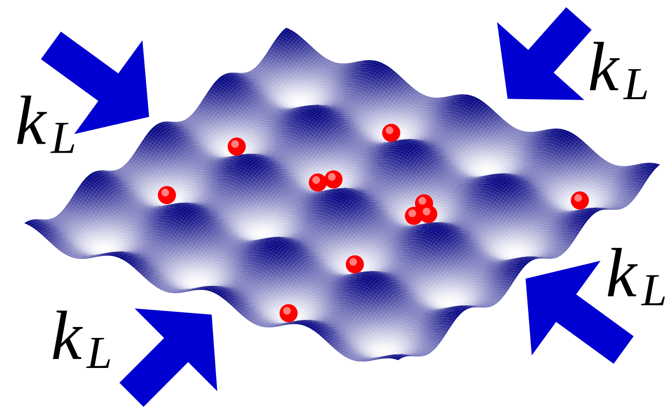
- Bose-Hubbard model
  - *Single particle in a lattice*
  - *Two-particle states in a lattice:* **dimers**
  - *Three-particle states in a lattice:* **trimers**
- Many-body physics of tightly-bound dimers
  - *Effective Hamiltonian*
  - *Repulsively bound dimers:* **droplets** (FM)
  - *Attractively bound dimers:* **checkerboard crystal** (AFM)

Bose-Hubbard Hamiltonian

Neutral bosons in tight-binding periodic potential

$$H = \sum_j \varepsilon_j \hat{n}_j - J \sum_{\langle j,i \rangle} b_j^\dagger b_i + \frac{U}{2} \sum_j \hat{n}_j (\hat{n}_j - 1)$$



[atoms in deep optical lattice]

$\varepsilon_j \ (= 0)$: single-particle energy

$b_j \ (b_j^\dagger)$: boson annihilation (creation) operator at site j

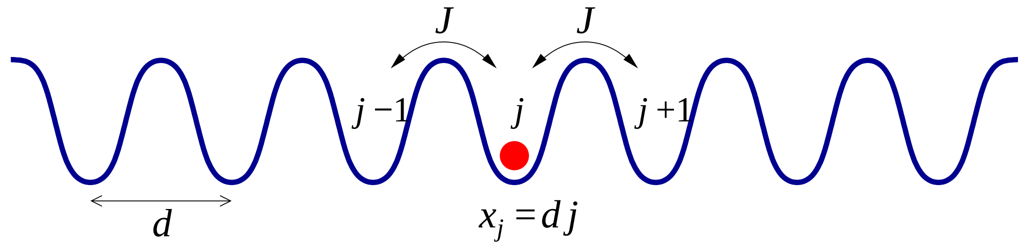
$\hat{n}_j \equiv b_j^\dagger b_j$: number operator

$J \ (> 0)$: inter-site tunneling

$U \propto a$: on-site interaction ($U > 0$ repulsion; $U < 0$ attraction)

Jaksch et al, Phys. Rev. Lett. **81**, 3108 (1998)

Single particle in a lattice (1D)



State vector

$$|\psi\rangle = \sum_j \psi(x_j)\,|x_j\rangle \quad \Downarrow$$

Difference equation

$$-J\left[\psi(x_{j-1}) + \psi(x_{j+1})\right] = E^{(1)}\,\psi(x_j)$$

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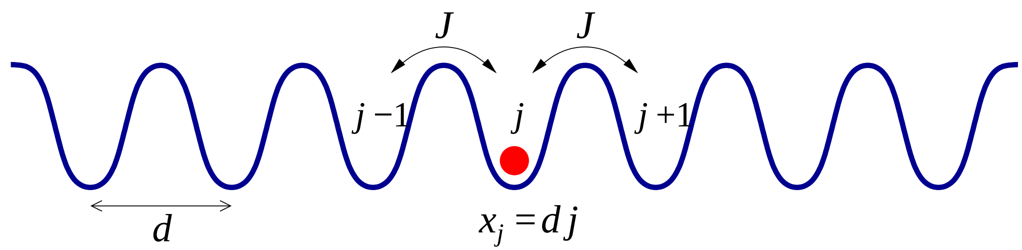

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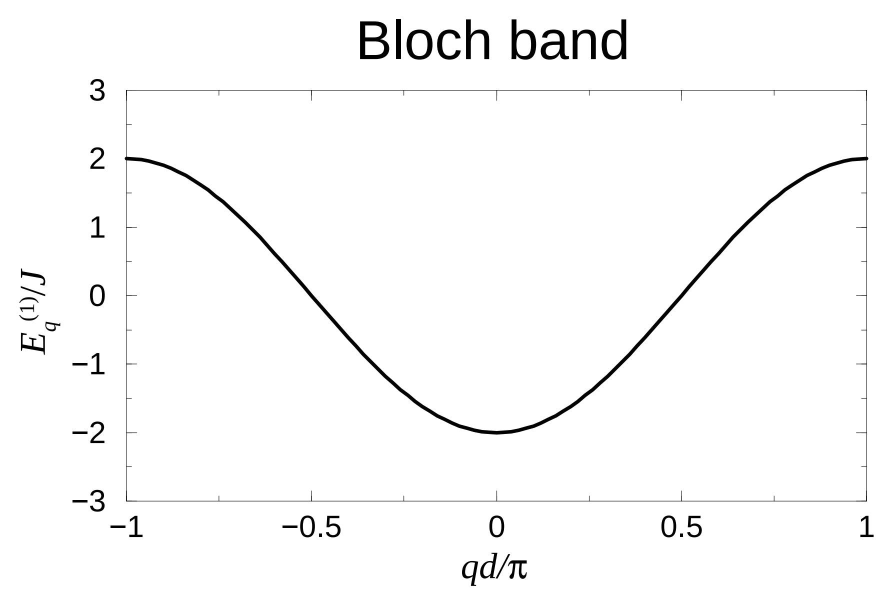
Solution

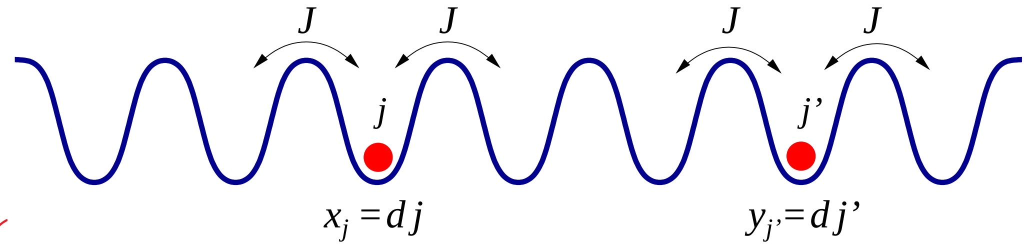
$$\psi_q(x_j) = e^{iqx_j}$$

Dispersion relation

$$E_q^{(1)} = -2J\cos(qd)$$



Two particles in Hubbard model (1D)



State vector

$$|\Psi\rangle = \sum_{j,j'} \Psi(x_j, y_{j'})\, |x_j, y_{j'}\rangle \;\Downarrow$$

Recurrence relation

$$-J\left[\Psi(x_{j-1}, y_{j'}) + \Psi(x_{j+1}, y_{j'}) + \Psi(x_j, y_{j'-1}) + \Psi(x_j, y_{j'+1})\right] + U\delta_{jj'}\Psi(x_j, y_{j'}) = E^{(2)}\,\Psi(x_j, y_{j'})$$

Valiente, Petrosyan, J. Phys. B **41**, 161002 (2008)

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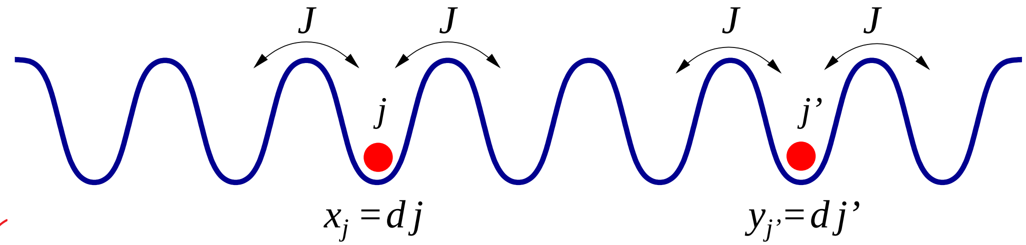

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$R = \frac{1}{2}(x+y)$ center of mass & $r = x - y$ relative coordinates $\Rightarrow$

Two-particle wavefunction (with K center-of-mass quasimomentum)

$$\Psi(x,y) = e^{iKR}\,\psi_K(r)$$

Recurrence relation (with $J_K \equiv 2J\cos(Kd/2)$ and $r_i = di$ ($i = j - j'$))

$$-J_K\left[\psi_K(r_{i-1}) + \psi_K(r_{i+1})\right] + U\delta_{r0}\psi_K(r_i) = E_K^{(2)}\psi_K(r_i)$$

Valiente, Petrosyan, J. Phys. B **41**, 161002 (2008)

Solution: Scattering states

Relative coordinate wavefunction

$$\psi_{K,k}(r_i) = \cos(k|r_i| + \delta_{K,k})$$

with $\delta_{K,k}$ scattering phase shift

$$\tan(\delta_{K,k}) = -\frac{U \csc(kd)}{4J \cos(Kd/2)}$$

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Generalized 1D scattering lengths

$$a_K = -\lim_{kd\to{0 \atop \pi}} \frac{\partial\delta_{K,k}}{\partial k} = \mp\frac{2J_K}{U}\,d$$

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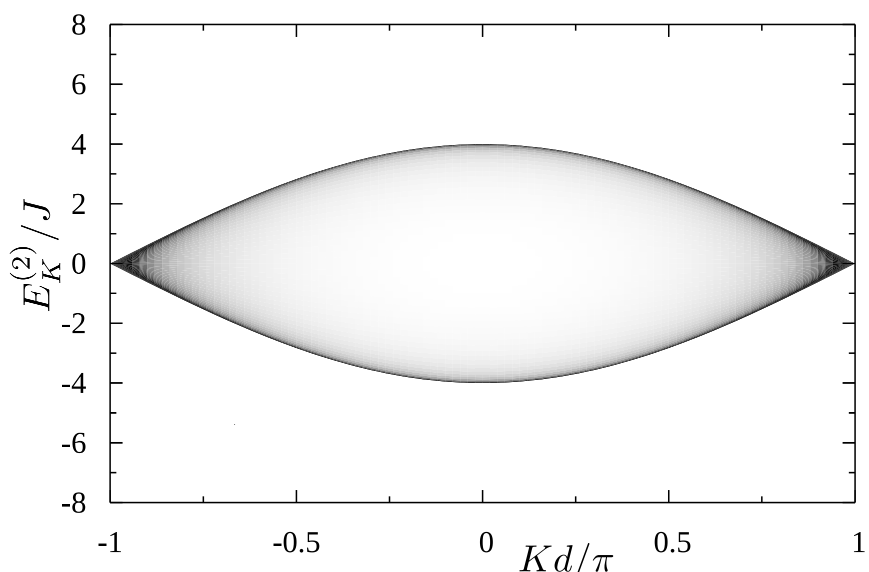
Spectrum of scattering states

$$E^{(2)}_{K,k} = -4J\cos(Kd/2)\cos(kd)$$

Density of states

$$\rho(E,K) \propto \frac{1}{\sqrt{[4J\cos(Kd/2)]^2 - E^2}}$$



Solution: Interaction-bound states

Repulsive interaction $U > 0$

Relative coordinate wavefunction

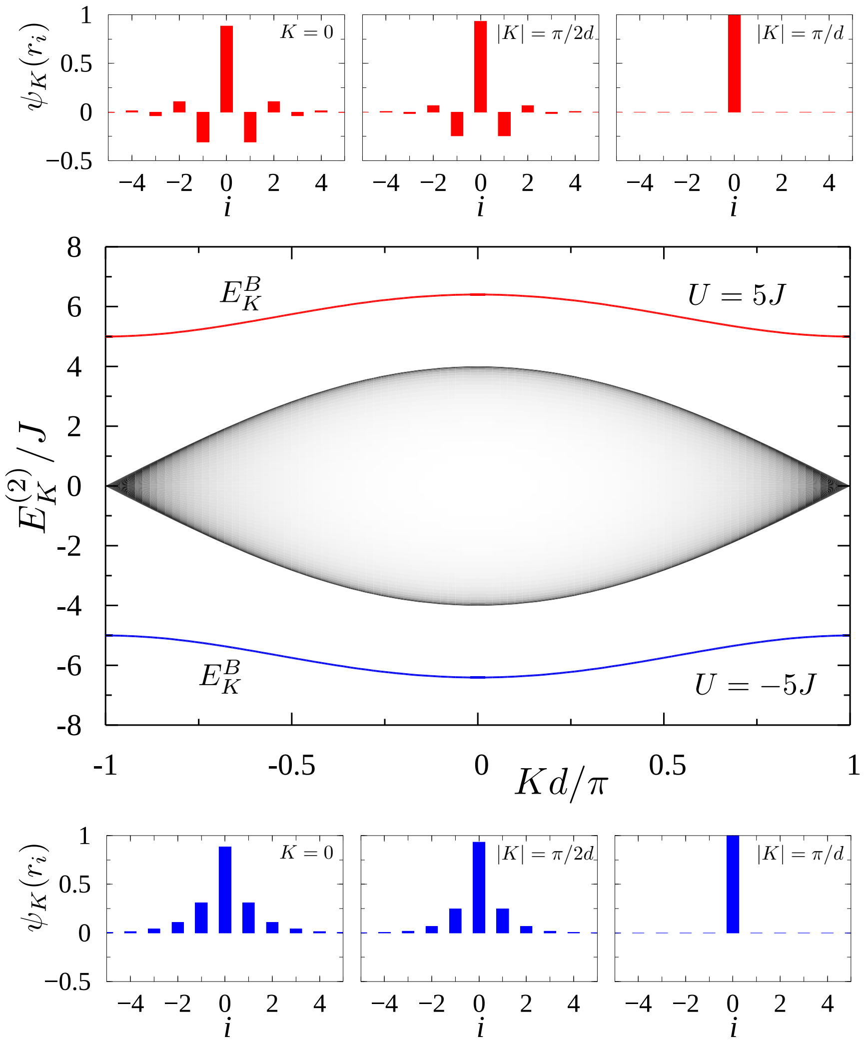
$$\psi_K(r_i) = \frac{\sqrt{\mathcal{U}_K}}{\sqrt[4]{\mathcal{U}_K^2+1}} \left(\mathcal{U}_K - \sqrt{\mathcal{U}_K^2+1}\right)^{|i|}$$

with $\mathcal{U}_K \equiv U/(2J_K)$ & $J_K \equiv 2J\cos(Kd/2)$

Dimer dispersion relation

$$E_K^B = \sqrt{U^2 + 4J_K^2} \quad \Rightarrow$$

- $E_{\pi/d}^B = |U| = U$
- $E_0^B = \sqrt{U^2 + 16J^2}$



Solution: Interaction-bound states

Attractive interaction $U < 0$

Relative coordinate wavefunction

$$\psi_K(r_i) = \frac{\sqrt{\mathcal{U}_K}}{\sqrt[4]{\mathcal{U}_K^2+1}} \left(\sqrt{\mathcal{U}_K^2+1} - |\mathcal{U}_K|\right)^{|i|}$$

with $\mathcal{U}_K \equiv (U/2J_K)$ & $J_K \equiv 2J\cos(Kd/2)$

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$$E_K^B = -\sqrt{U^2 + 4J_K^2} \quad \Rightarrow$$

- $E_0^B = -\sqrt{U^2+16J^2}$
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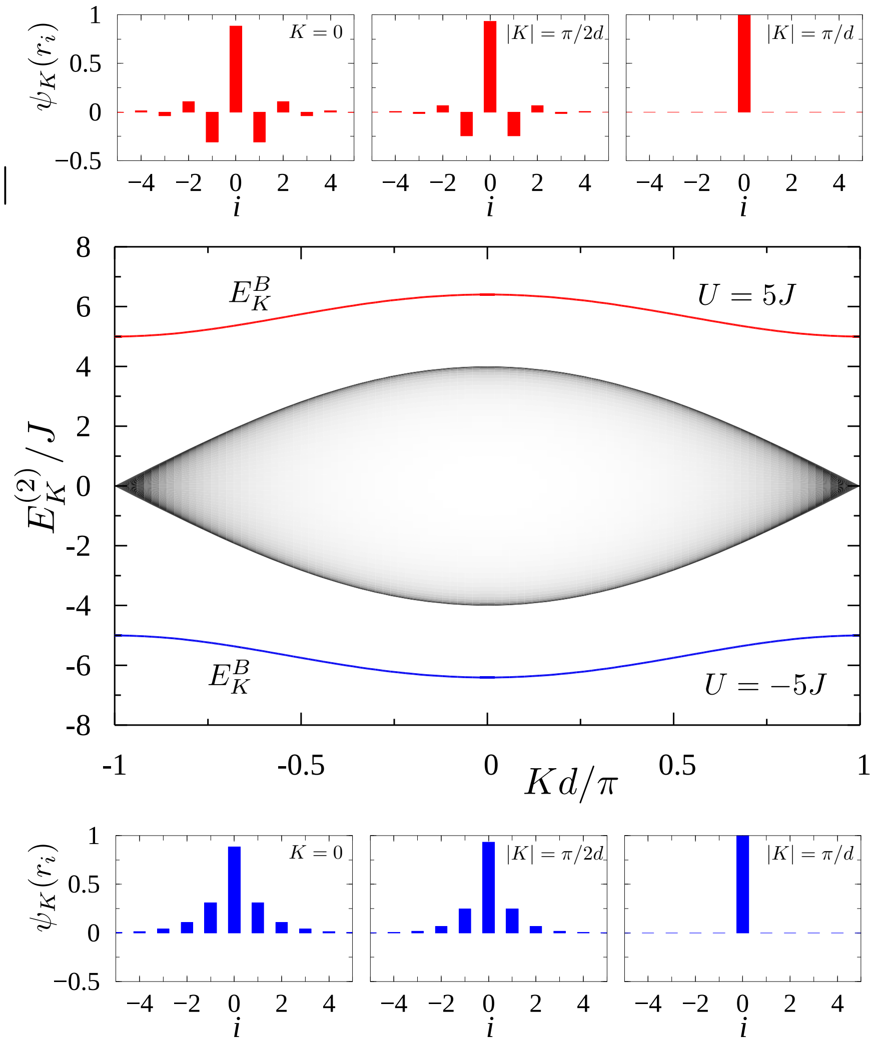


Solution: Interaction-bound states

Strong interaction $|U| > J$

Relative coordinate wavefunction

$$\psi_K(r_i) \simeq \sqrt{\frac{U^2 - J_K^2}{U^2 + J_K^2}} \left(- \frac{J_K}{U} \right)^{|i|} \quad \Rightarrow$$

localization length $\zeta \leq [2 \ln(U/2J)]^{-1}$

$\zeta < 1$ for $U/J > 2\sqrt{e} \Rightarrow$

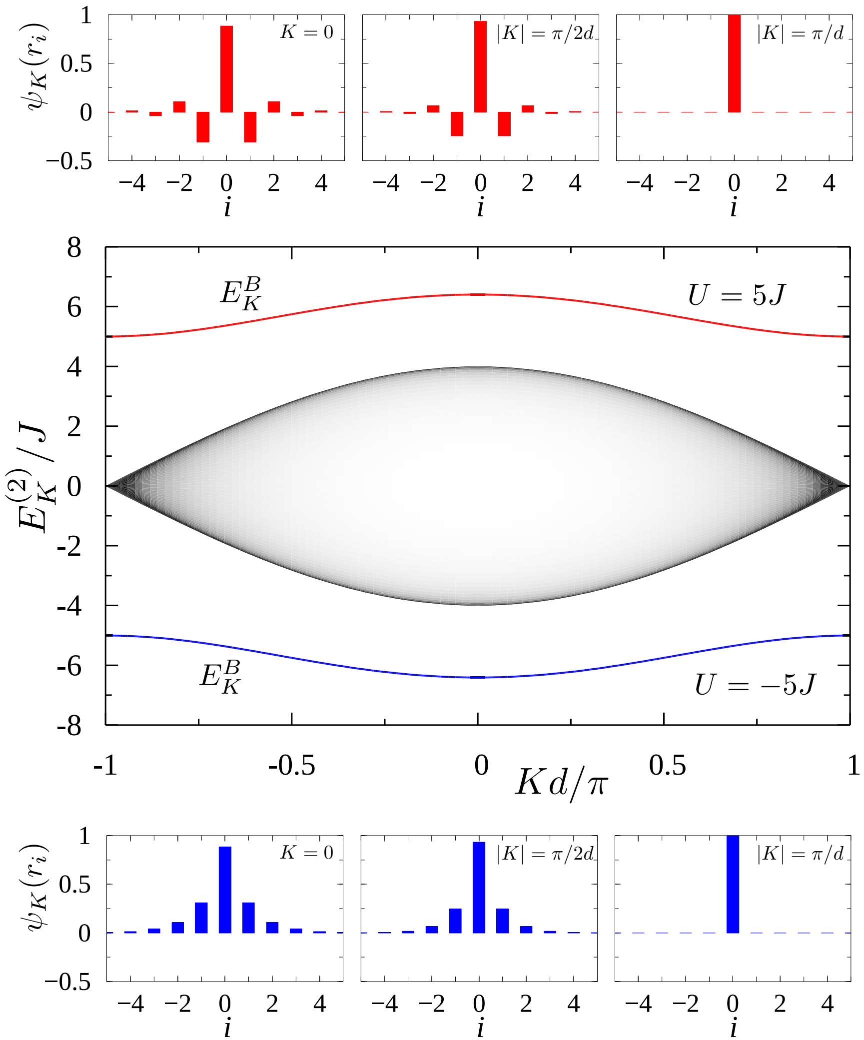
Tightly-bound dimer

Dimer dispersion relation

$$E_K^B \simeq (U - 2\tilde{J}) - 2\tilde{J} \cos(Kd)$$

with $(U - 2\tilde{J})$ dimer "internal" energy

$\tilde{J} \equiv -2J^2/U$ effective tunneling rate



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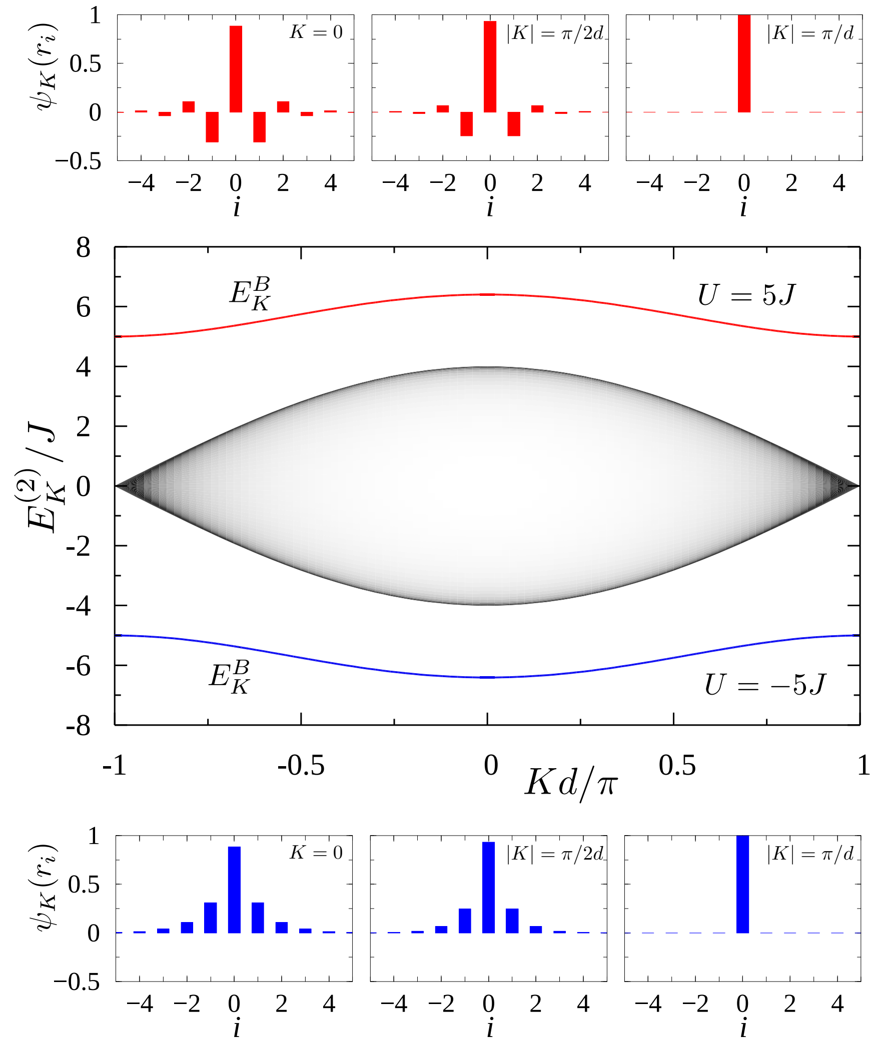

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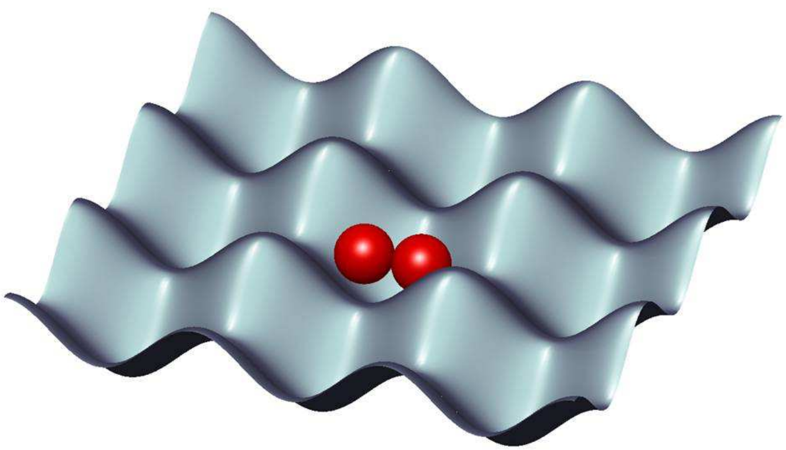
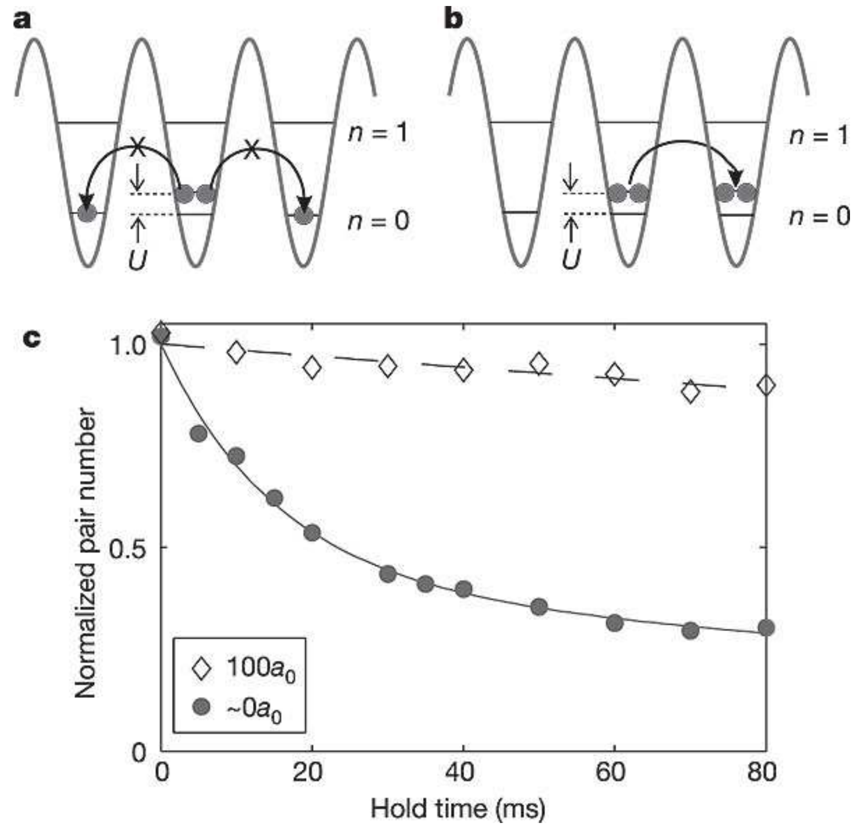
Effective dimer Hamiltonian

$$H_{\text{eff}} = (U - 2\tilde{J}) \sum_j \hat{m}_j - \tilde{J} \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Repulsively bound atom pair: Experiment

Single dimer

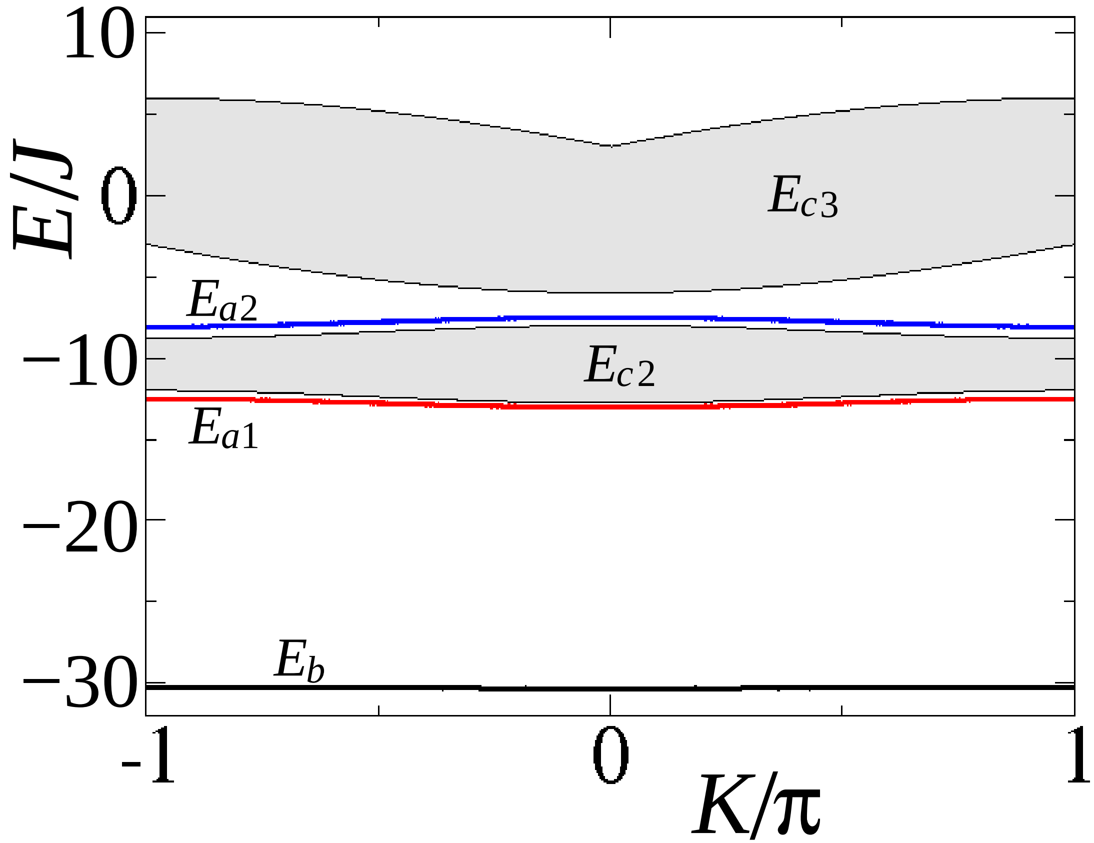


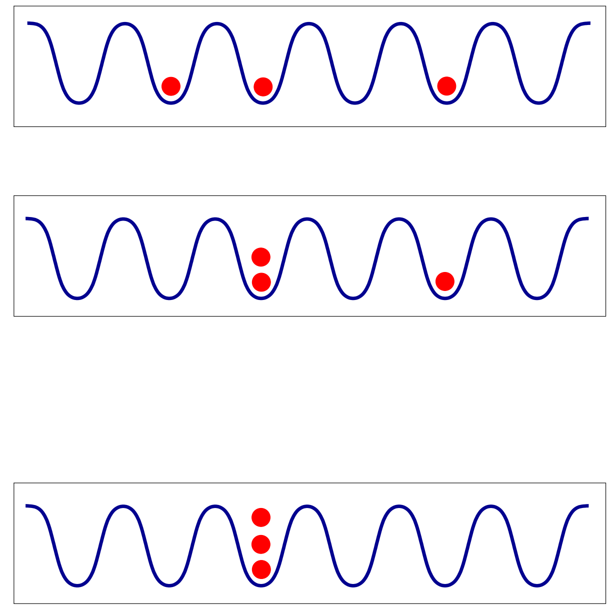


Winkler et al, Nature **441**, 853 (2006)

Three particles in Hubbard model (1D)

Complete three-body spectrum $[U = -10J]$





Valiente, Petrosyan, Saenz, Phys. Rev. A **81**, 011601(R) (2010)

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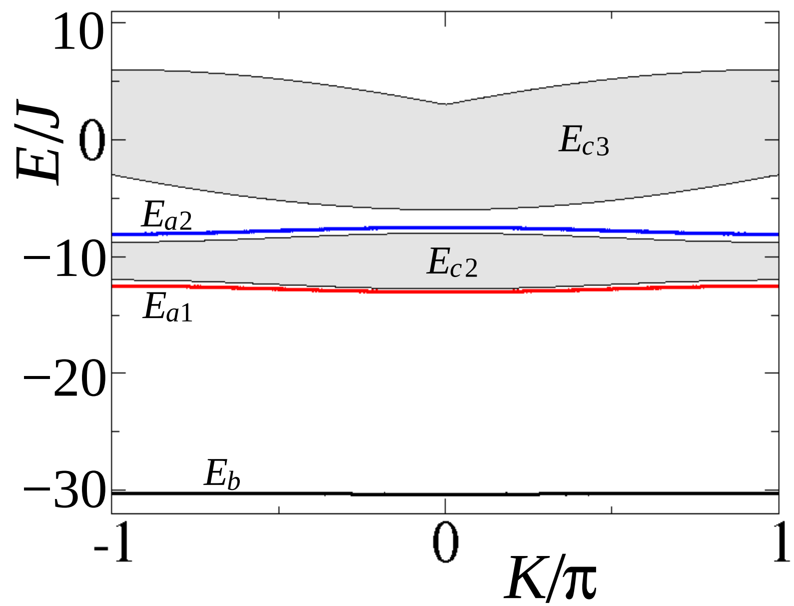
Three-body continuum

$$E_{c3} = \epsilon(k_1) + \epsilon(k_2) + \epsilon(K - k_1 - k_2)$$

$$\epsilon(k) = -2J\cos(k) \qquad K = k_1 + k_2 + k_3$$



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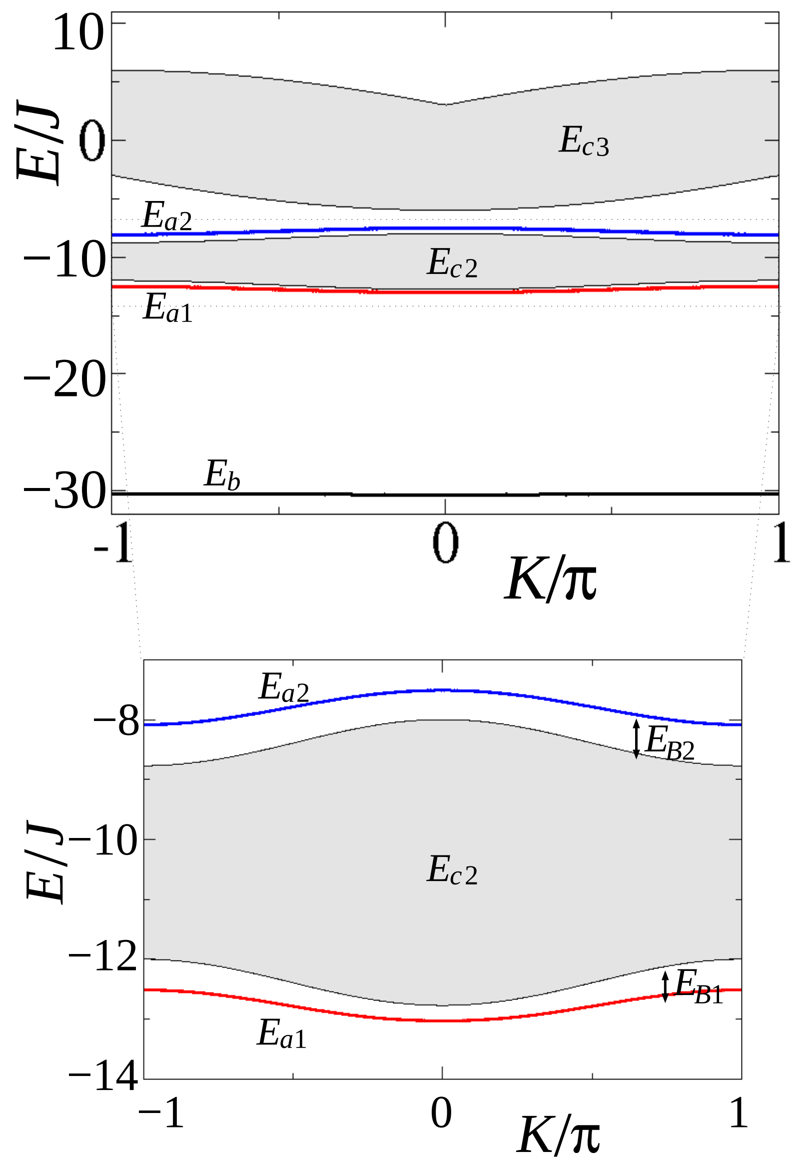
Two-body continuum

$$E_{c2} = \epsilon^{(2)}(Q) + \epsilon(K - Q)$$

$$\epsilon^{(2)}(Q) = \mathrm{sgn}(U)\sqrt{U^2 + [4J\cos(Q/2)]^2}$$
$$\simeq (U - 2\tilde{J}) - 2\tilde{J}\cos(Q)$$



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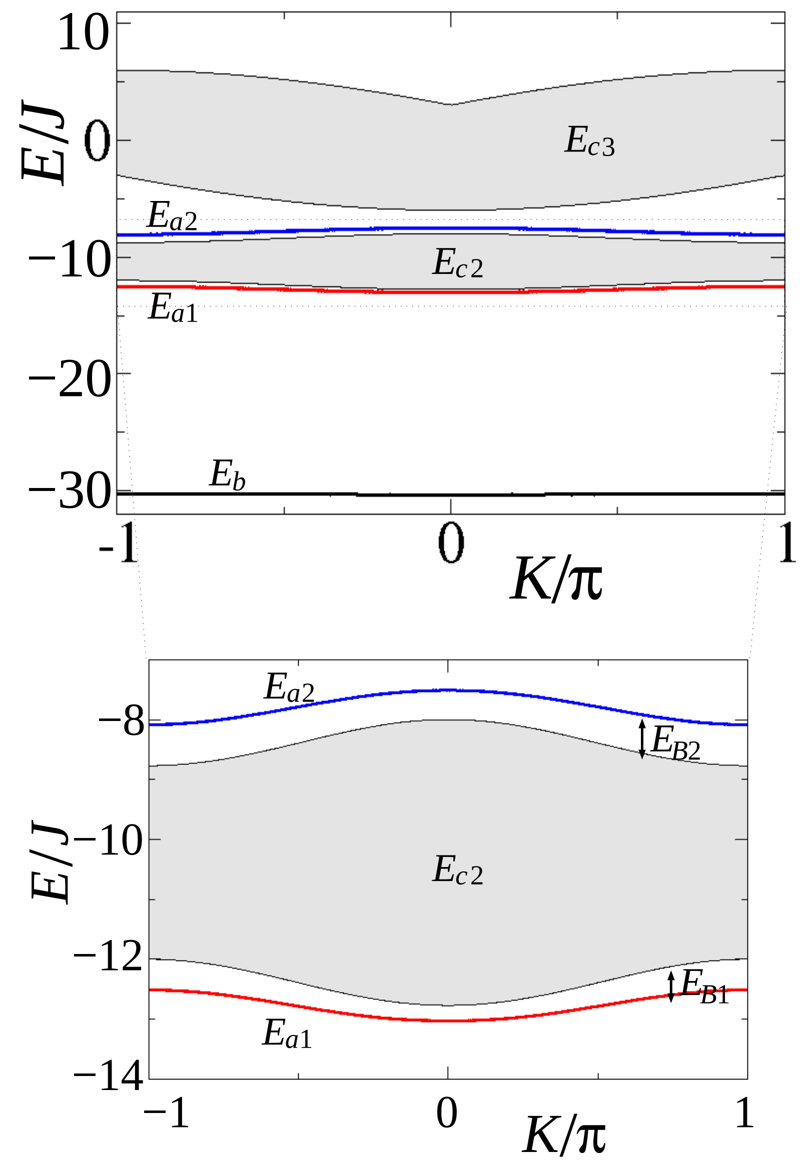
Weakly-bound (off-site) trimers

$$E_{a1(2)} \simeq U + O(J)$$



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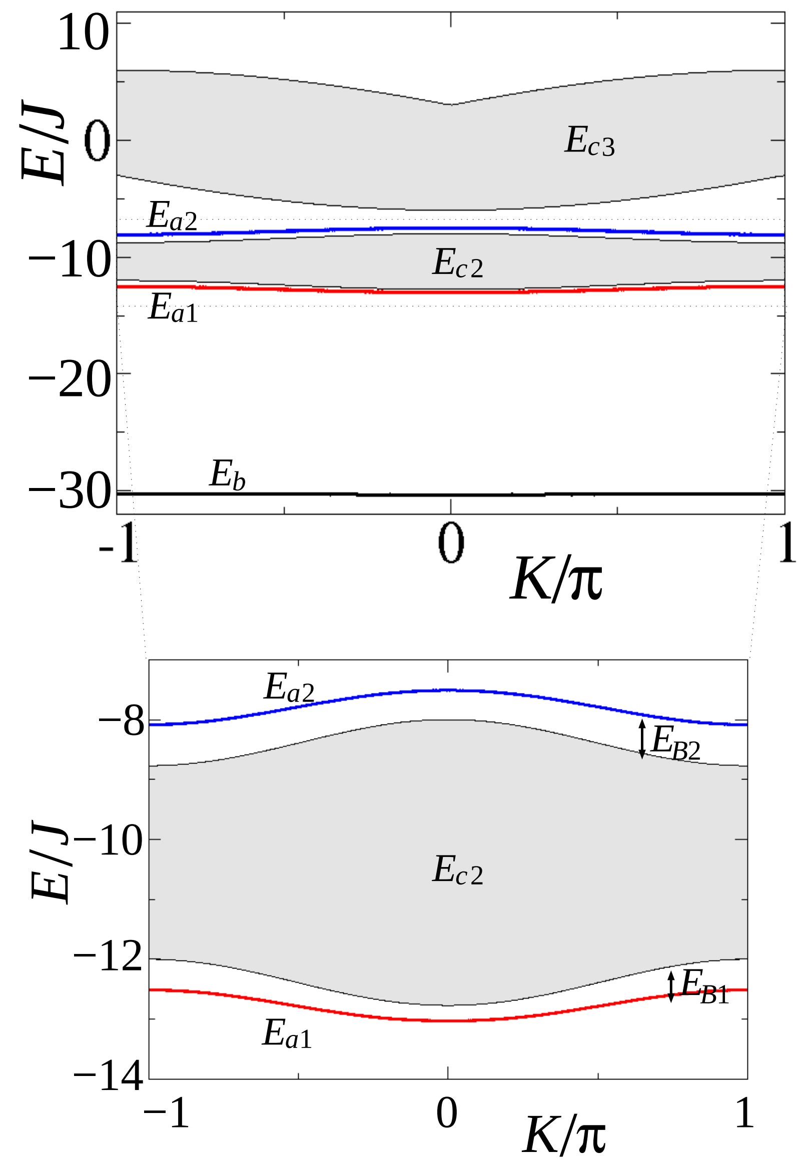
Weakly-bound (off-site) trimers

$$E_{a1(2)} \simeq U + O(J)$$

Strongly-bound (on-site) trimer

$$E_b \simeq 3U$$



Formalism: Three-body bound states

State vector in momentum representation

$$|\psi\rangle = \frac{1}{(2\pi)^{3/2}} \iiint_{\Omega^3} dk_1 dk_2 dk_3 \, \psi(k_1, k_2, k_3) \, |k_1, k_2, k_3\rangle \qquad k_j \in \Omega \equiv [-\pi, \pi]$$

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$$H|\psi\rangle = E|\psi\rangle \quad \Rightarrow \quad \boxed{\psi(k_1, k_2, k_3) = -\frac{M(k_1)+M(k_2)+M(k_3)}{\epsilon(k_1)+\epsilon(k_2)+\epsilon(k_3)-E}}$$

with

$$M(k)[1 + I_E(k)] = -\frac{U}{\pi} \int_{-\pi}^{\pi} dq \, \frac{M(q)}{\epsilon(k)+\epsilon(q)+\epsilon(K-k-q)-E}$$

$$I_E(k) \equiv \frac{U}{2\pi} \int_{-\pi}^{\pi} dq \, \frac{1}{\epsilon(k)+\epsilon(q)+\epsilon(K-k-q)-E} = -\frac{\mathrm{sgn}[E-\epsilon(k)]U}{\sqrt{[E-\epsilon(k)]^2 - 16J^2 \cos^2[(K-k)/2]}}$$

Mattis, Rev. Mod. Phys. **58**, 361 (1986)

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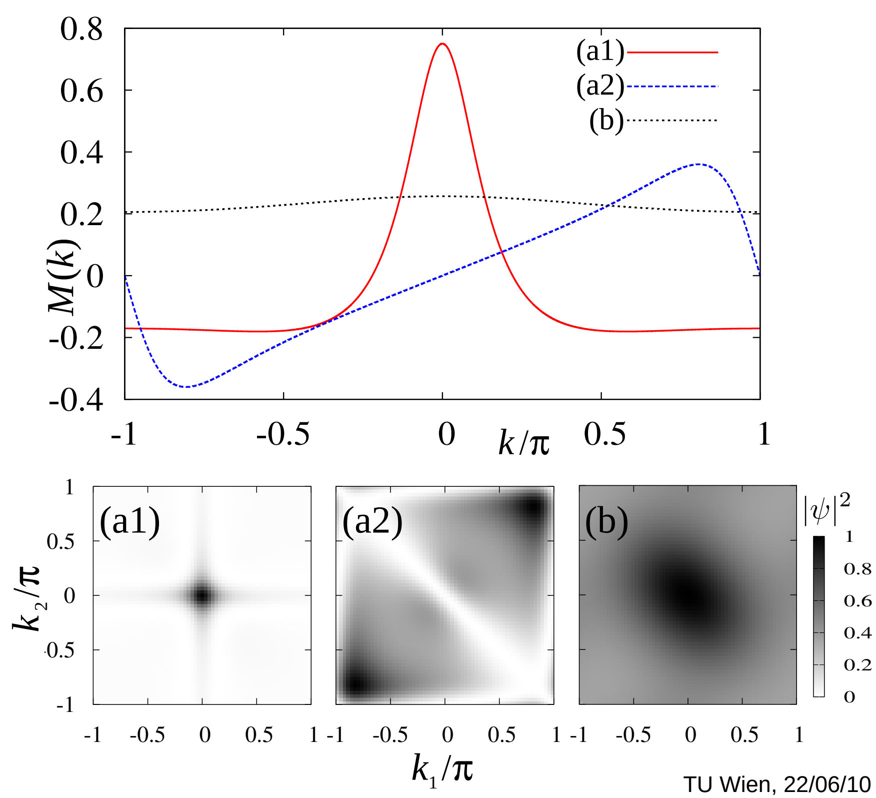

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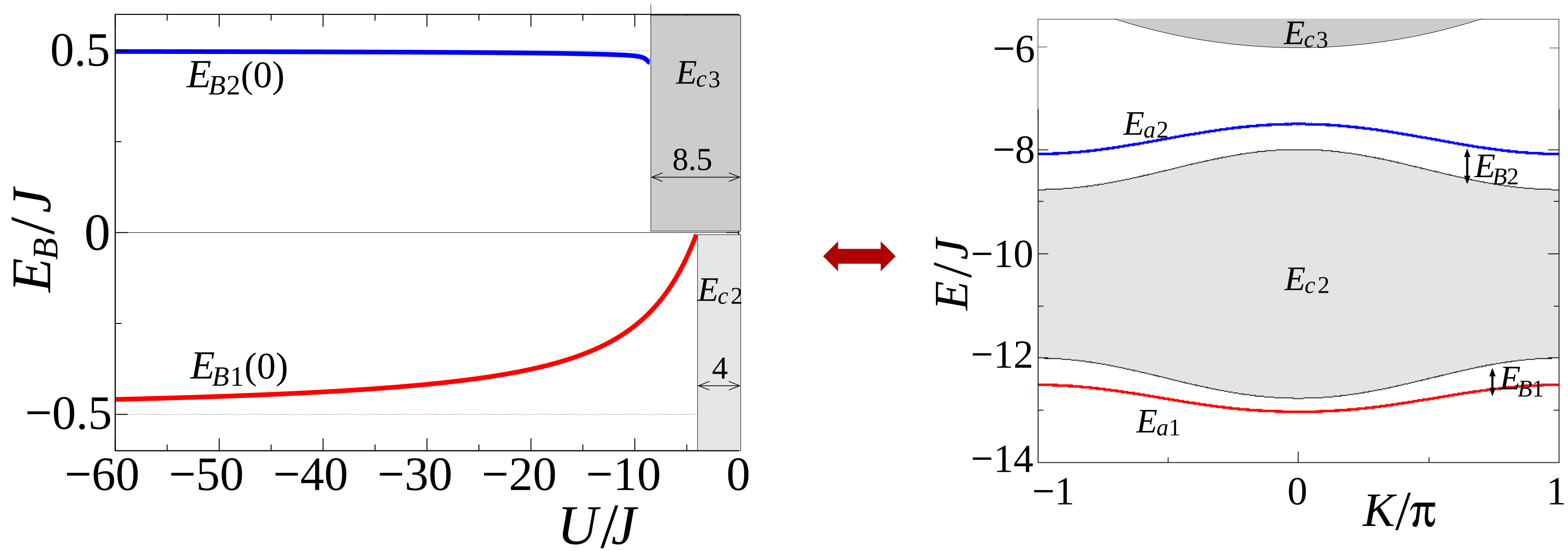
Thresholds & limits of *U*

Binding energies



$$E_{B1}(K) = E_{a1}(K) - E_{c2}(K, Q = 0)$$

$$E_{B2}(K) = E_{a2}(K) - E_{c2}(K, Q = \pi)$$

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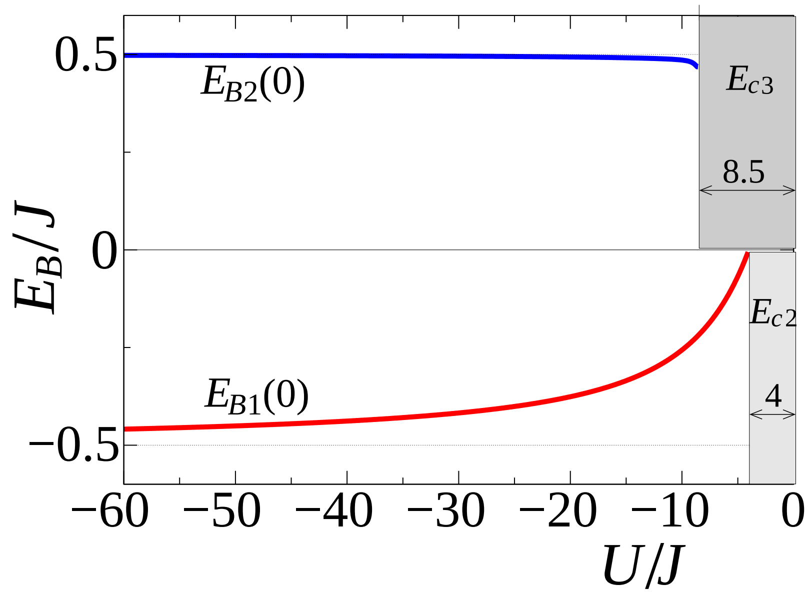

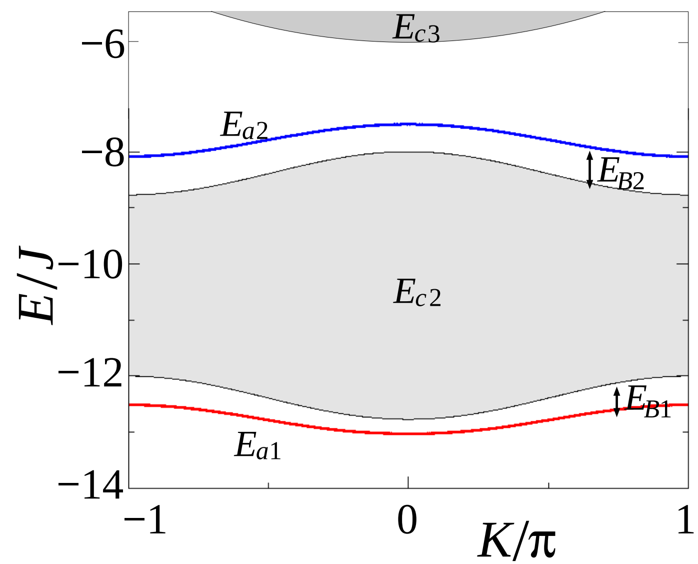

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For $|U| \lesssim 4J$ $\Rightarrow$ $E_{a1} \to E_{c2}$ ($E_{B1} \to 0$: Scattering resonance)

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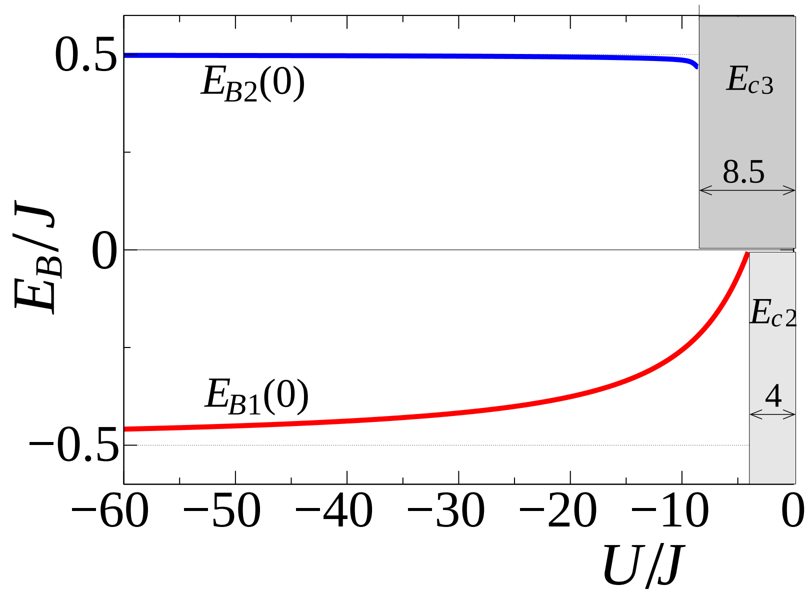

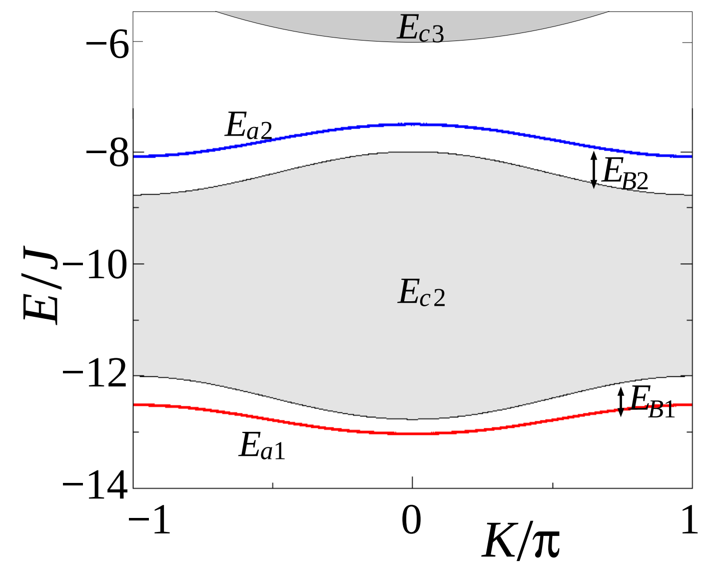

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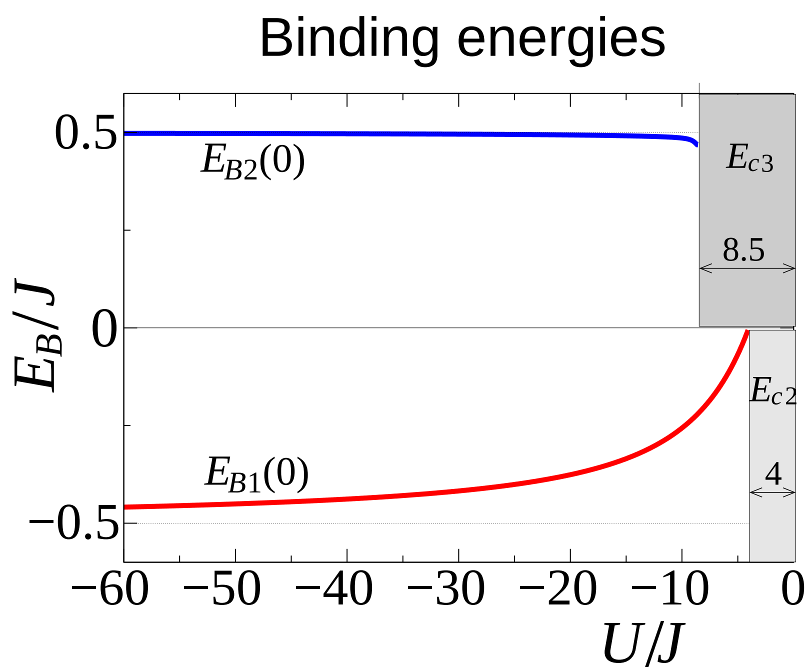
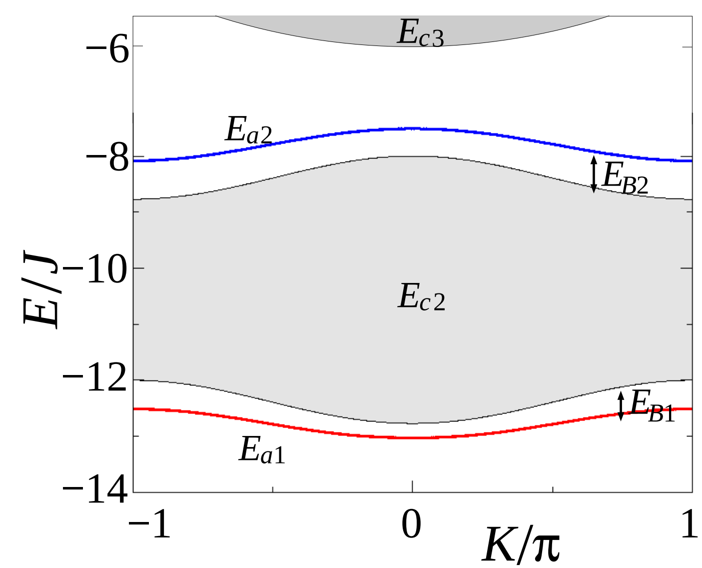
For $|U| \lesssim 8.5J \quad \Rightarrow \quad E_{a2} \in E_{c3}$ (for $|U| \leq 8J \;\; E_{c2} \bigcap E_{c3} \neq 0$)

Thresholds & limits of U





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- $\lim_{U \to -\infty} E_{B1} = -\frac{J}{2}$
- $\lim_{U \to -\infty} E_{B2} = \frac{J}{2}$

Effective Hamiltonian for dimer+monomer

$\boxed{H_{\rm eff} = H_1 + H_2 + H_{\rm int}}$ for $|U| > 8J$ $(E_{c2} \bigcap E_{c3} = 0)$

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Monomer (single-particle) Hamiltonian

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$$H_2 = (U - 2\tilde{J}) \sum_j \hat{m}_j - \tilde{J} \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Dimer-monomer interaction

$$H_{\rm int} = \tilde{V} \sum_j \hat{m}_j \hat{n}_{j\pm 1} - W \sum_j (c_{j+1}^\dagger b_j^\dagger c_j b_{j+1} + c_j^\dagger b_{j+1}^\dagger c_{j+1} b_j)$$

$\tilde{V} = -7J^2/2U$: nearest-neighbor interaction

$W = 2J$: (particle) exchange interaction

Effective Hamiltonian for dimer+monomer

$$\boxed{H_{\text{eff}} = H_1 + H_2 + H_{\text{int}}}$$ for $|U| > 8J$ $(E_{c2} \bigcap E_{c3} = 0)$

Monomer (single-particle) Hamiltonian

$$H_1 = -J \sum_j (b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j)$$

Dimer (bound pair) Hamiltonian

$$H_2 = (U - 2\tilde{J}) \sum_j \hat{m}_j - \tilde{J} \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

Dimer-monomer interaction

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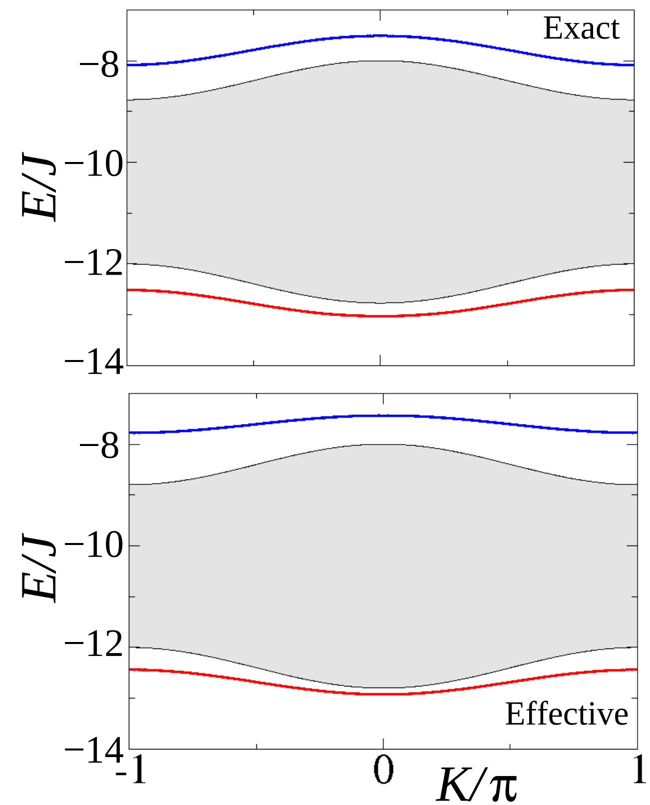
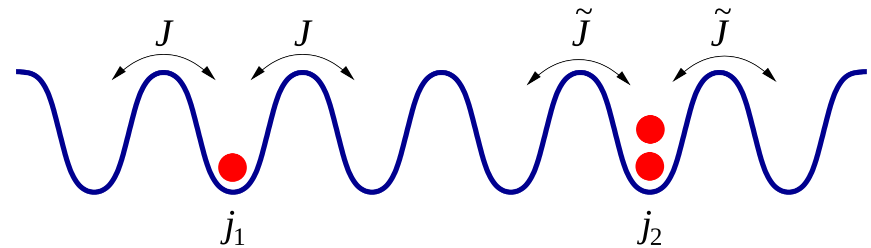
dimer-monomer spectrum



Solution: Bound states ($K = 0, \pm\pi$)

State vector



$$|\Psi\rangle = \sum_{j_1 \neq j_2} \Psi(j_1, j_2)\, |j_1, j_2\rangle$$

with

$$\Psi(j_1, j_2) = e^{iK(j_1+j_2)/2} e^{-i\delta_K j_r} \phi_K(j_r) \qquad (j_r \equiv j_1 - j_2 \quad \tan(\delta_K) = \tan(\tfrac{K}{2}) \tfrac{J-\tilde{J}}{J+\tilde{J}})$$

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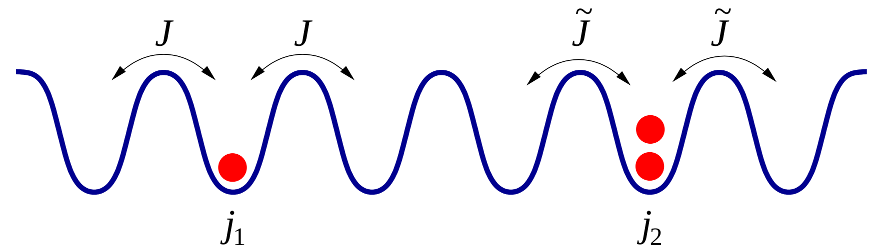

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Recurrence relations for relative coordinate wavefunction ($|j_r| > 1$)

$$\phi_K(0) = 0$$

$$\bar{J}_K[\phi_K(j_r + 1) + \phi_K(j_r - 1)] + \bar{E}\phi_K(j_r) = 0$$

$$\bar{J}_K \phi_K(\pm 2) + W_K \phi_K(\mp 1) + [\bar{E} - \tilde{V}]\phi_K(\pm 1) = 0$$

with $\bar{J}_K \equiv \sqrt{J^2 + \tilde{J}^2 + 2J\tilde{J}\cos(K)}$ $\quad W_K \equiv W\cos(K)$ $\quad \bar{E} \equiv E - (U - 2\tilde{J})$

$\bar{J}_{0,\pi} = J \pm \tilde{J}$

Solution: Bound states ($K = 0, \pm\pi$)

Exponential ansatz $\phi_K(j_r > 0) \propto \alpha_K^{j_r - 1}$

$\& \ \phi_K(-j_r) = \pm\phi_K(j_r)$ ("$+$" symmetric (triplet); "$-$" antisymmetric (singlet) solutions)

$\Rightarrow \quad \alpha_K^{(\pm)} = -\frac{\bar{J}_K}{\tilde{V} \mp W_K}$ with $\bar{E}_{a1(2)} = -\frac{\bar{J}_K[1+(\alpha_K^{(\pm)})^2]}{\alpha_K^{(\pm)}}$

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Bound states ($|\alpha_K^{(\pm)}| < 1$)

- $|\tilde{V} \mp W_K| > \bar{J}_K \quad \Rightarrow \quad |\tilde{V} \mp 2J| > J \pm \tilde{J}$

  For $U \gg J \quad E_{B1(2)} = \bar{E}_{a1(2)} \mp 2\bar{J}_K \simeq \mp \frac{J}{2}$ ($\tilde{V} \ll J$)

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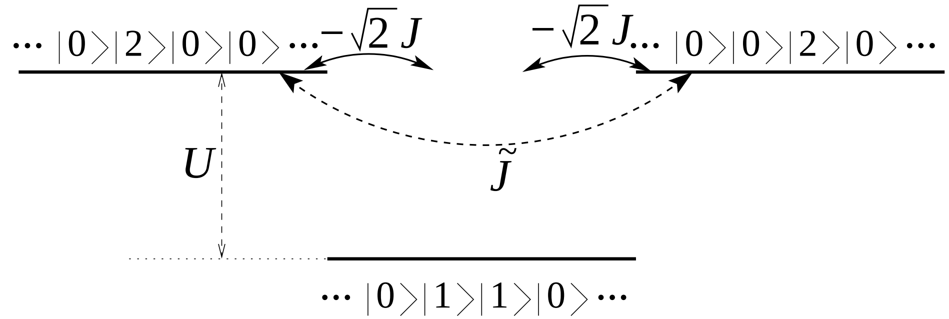
$\Rightarrow$ Exchange interaction $W = 2J$ binds dimer and monomer

Many-body physics of tightly-bound dimers

Interaction-bound atom pair—Dimer

Energies of $|2,0\rangle$ & $|0,2\rangle$ are larger (smaller) than of $|1,1\rangle$ by U



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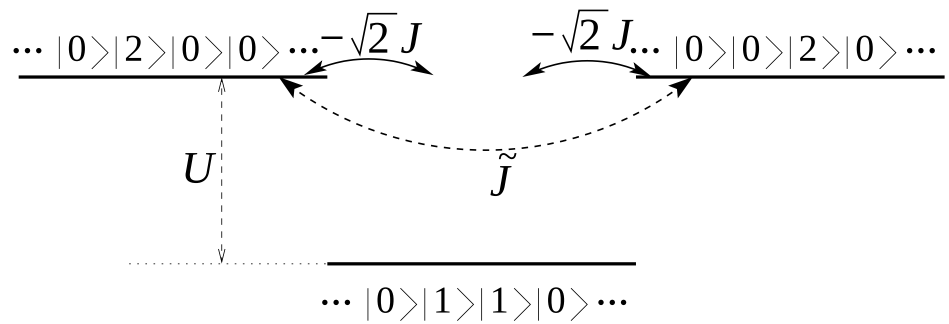

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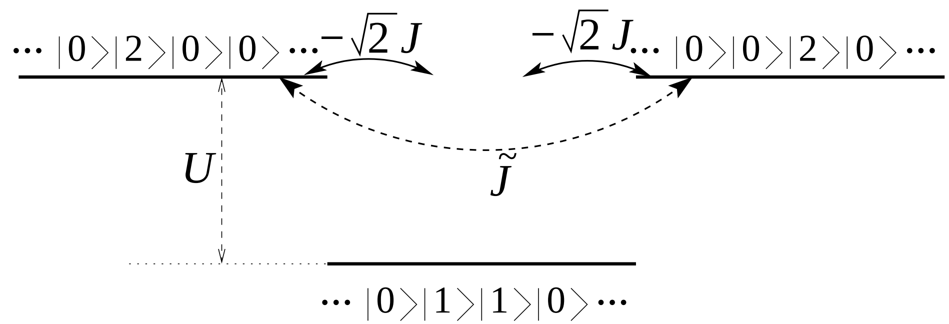

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On-site interaction (repulsion $U > 0$) binds two atoms into a **dimer**

But $|2,0\rangle \to |1,1\rangle \to |0,2\rangle$ is resonant (second order in J)

Effective tunneling rate for **dimer** $|2,0\rangle \to |0,2\rangle$ is $\tilde{J} = -\frac{2J^2}{U}$

Slow dynamics ($|\tilde{J}| \ll J$)

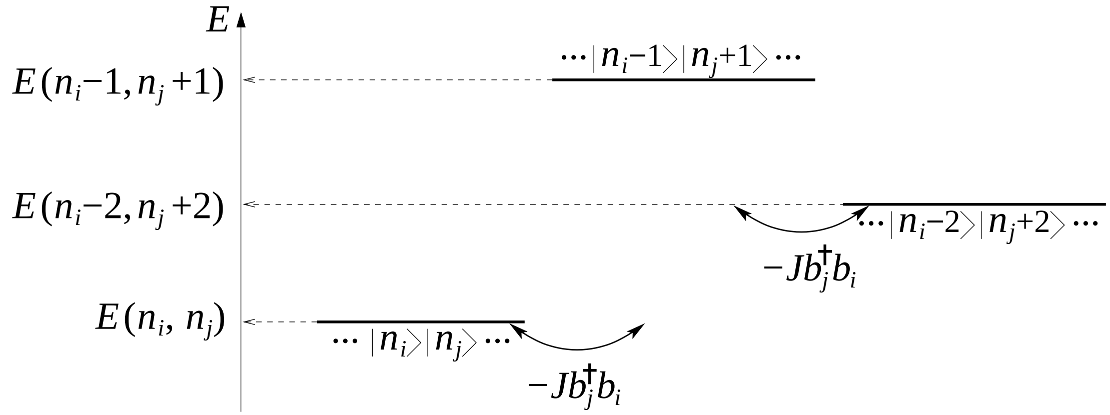
Effective Hamiltonian for dimers ($|U| \gg J$)

Define

$$c_j = \frac{1}{\sqrt{2(\hat{n}_j + 1)}} \, b_j^2, \quad c_j^\dagger$$

$$\hat{m}_j = c_j^\dagger c_j$$

$$\tilde{J} = -\frac{2J^2}{U}$$



Adiabatic elimination of nonresonant states $|n_i \pm 1\rangle \, |n_j \mp 1\rangle \quad (n_l = 2m_l)$

$$\Downarrow$$

Petrosyan, Schmidt, Anglin, Fleischhauer, Phys. Rev. A **76**, 033606 (2007)

Effective Hamiltonian for dimers ($|U| \gg J$)

Effective Hamiltonian for paired bosons in a periodic potential

$$H_{\text{eff}} = 2\varepsilon \sum_j \hat{m}_j + U \sum_j \hat{m}_j \left(2\hat{m}_j - 1\right) - \tilde{J} \sum_{\langle j,i \rangle} c_j^\dagger \, \hat{T}\left(\hat{m}_j, \hat{m}_i\right) c_i + \tilde{J} \sum_{\langle j,i \rangle} \hat{S}\left(\hat{m}_j, \hat{m}_i\right)$$

Kinetic Energy $\hat{T}\left(\hat{m}_j, \hat{m}_i\right) = \delta_{\hat{m}_i \hat{m}_j} \sqrt{\left(2\hat{m}_j + 1\right)\left(2\hat{m}_i + 1\right)}$

Potential Energy $\hat{S}\left(\hat{m}_j, \hat{m}_i\right) + \hat{S}\left(\hat{m}_i, \hat{m}_j\right) = \frac{2\hat{m}_j^2 + 2\hat{m}_i^2 + \hat{m}_j + \hat{m}_i}{1 - 4\left(\hat{m}_j - \hat{m}_i\right)^2}$

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- Nearest-neighbor attraction/repulsion $>$ tunneling

$$\frac{\text{Kin.En}}{\text{Pot.En}} = \frac{3(m+1)(2m+1)}{8(4m^2 + 6m + 3)} < 0.2 \qquad \text{(1D)}$$

Petrosyan, Schmidt, Anglin, Fleischhauer, Phys. Rev. A **76**, 033606 (2007)

Repulsively-bound dimers ($U > 0$)

Nearest-neighbor interaction (potential) energy

$$\tilde{J}\,\frac{2\hat{m}_j^2 + 2\hat{m}_i^2 + \hat{m}_j + \hat{m}_i}{1 - 4\left(\hat{m}_j - \hat{m}_i\right)^2} \qquad \tilde{J} = -\frac{2J^2}{U} < 0$$

- For $m_i = m_j \rightarrow$ Attraction   For $m_i \neq m_j \rightarrow$ Repulsion

$\Rightarrow$ Dimer clustering into “droplets” with uniform filling!

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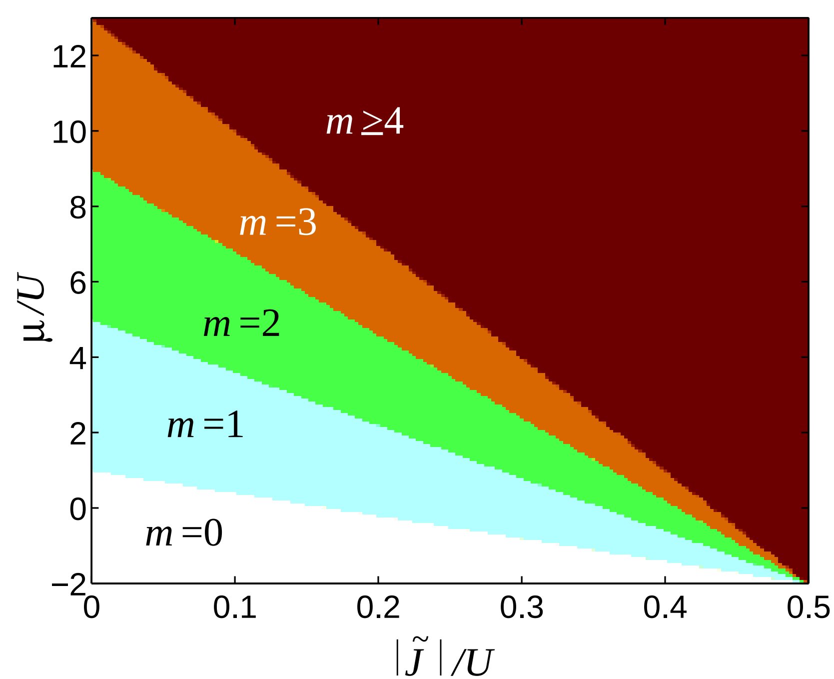
1D Phase Diagram $[\mu - \tilde{J}]$

Grand canonical ensemble

H_{eff} with uniform chem. potential

$$\boxed{\mu = -2\varepsilon}$$

Exacts diagonalization for 5 sites ($0 \leq m \leq 4$)



Only uniform commensurate filling (incompressible phases)

Single dimers per site ($U \gtrless 0$)

Effective Hamiltonian ($m = 0, 1 \ \forall\, j$)

$$H_{\text{eff}}^{(0,1)} = \left[2\varepsilon + U - 2d\tilde{J}\right] \sum_{j} \hat{m}_j - \tilde{J} \sum_{\langle j,i \rangle} c_j^{\dagger} c_i + 4\tilde{J} \sum_{\langle j,i \rangle} \hat{m}_j \hat{m}_i$$

Similar to Extended Hubbard Model (nearest-neighbor interaction)

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Equivalent spin-$\frac{1}{2}$ XXZ model Hamiltonian ($|0_j\rangle \to |\downarrow_j\rangle,\ |1_j\rangle \to |\uparrow_j\rangle$)

$$H_{XXZ} = 2h_z \sum_j \sigma_j^z - \tfrac{1}{4}\tilde{J} \sum_{\langle j,i \rangle} \left(\sigma_j^x \sigma_i^x + \sigma_j^y \sigma_i^y\right) + \tilde{J} \sum_{\langle j,i \rangle} \sigma_j^z \sigma_i^z$$

$h_z = \frac{1}{4}\left[2\varepsilon + U - 2d\tilde{J}\right] + 2d\tilde{J}$ – effective “magnetic field”

$\langle \sigma^z \rangle = [2\langle \hat{m} \rangle - 1]$ – fixed “magnetization” (averaged)

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Since $\frac{1}{4} < 1 \Rightarrow \quad H_{XXZ} \simeq H_{\rm Ising} = 2h_z \sum_j \sigma_j^z + \tilde{J} \sum_{\langle j,i \rangle} \sigma_j^z \sigma_i^z$

Droplets in a lattice ($U > 0$)

Strong dimer-dimer attraction ($8\tilde{J} > \tilde{J}$)

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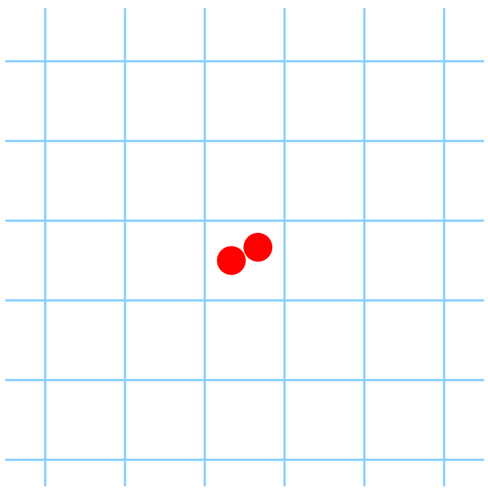

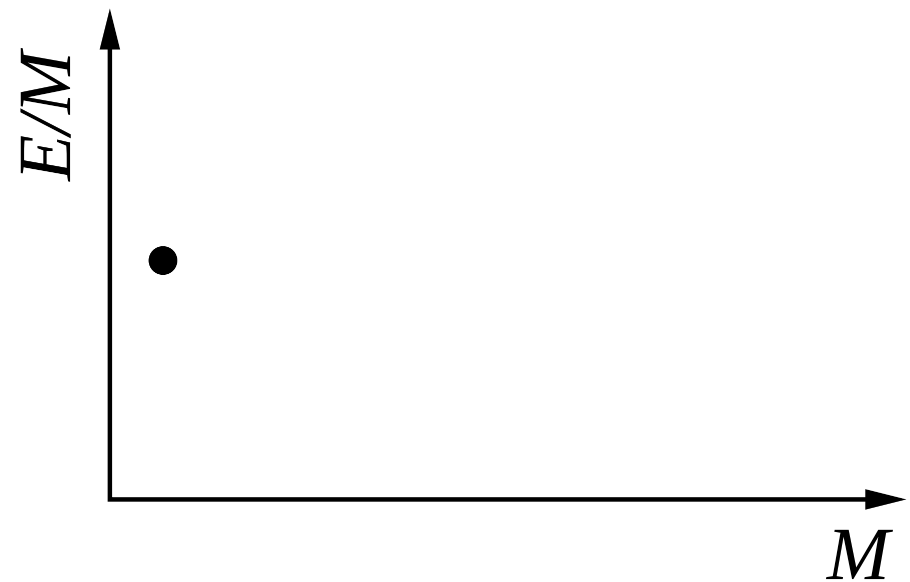

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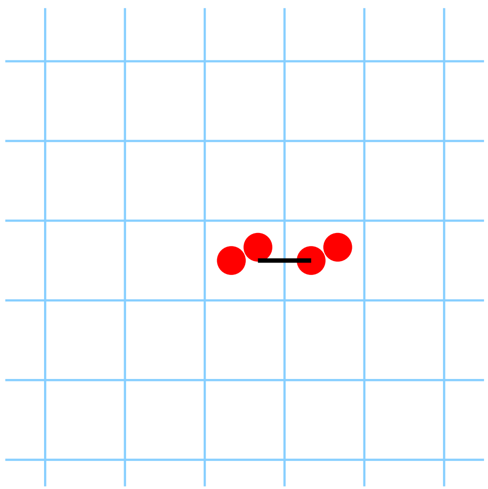

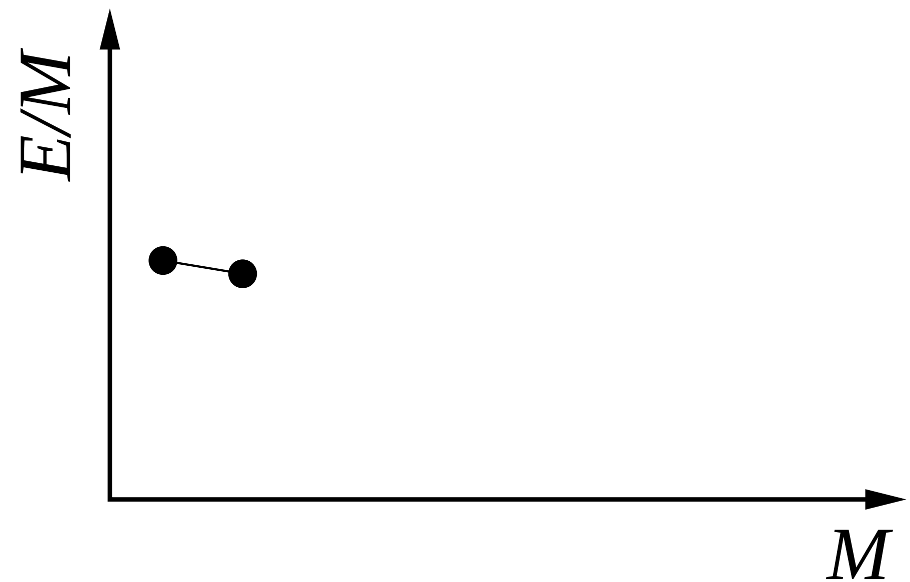

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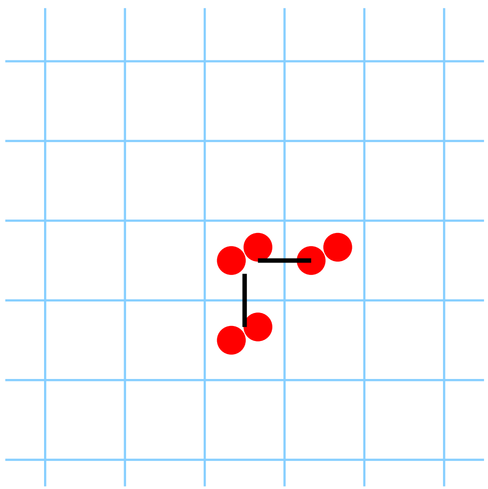

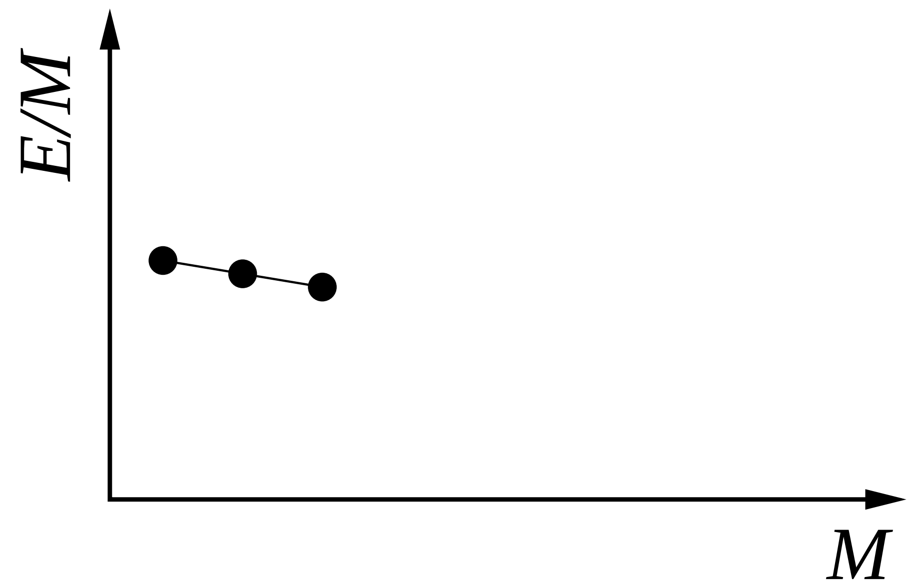

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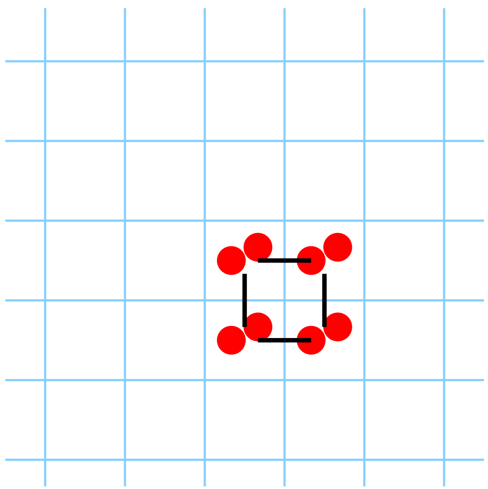

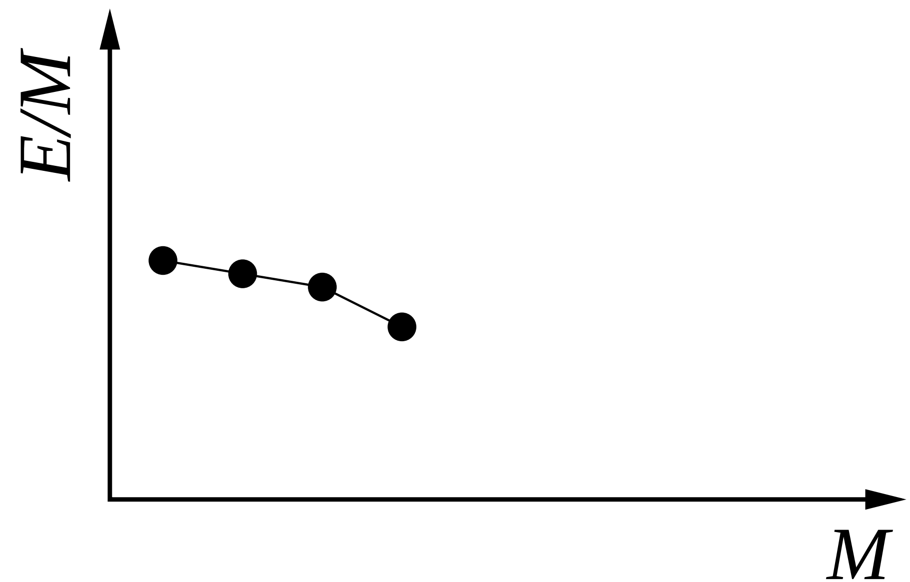

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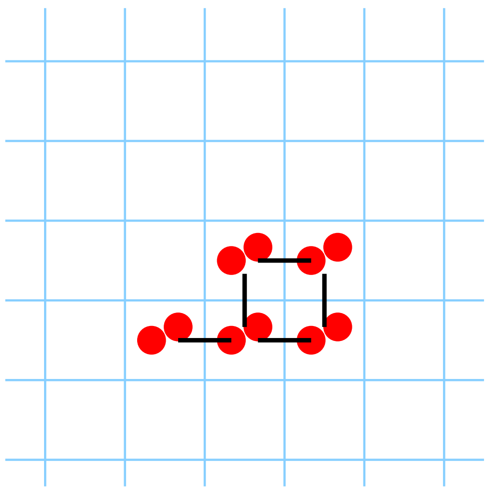

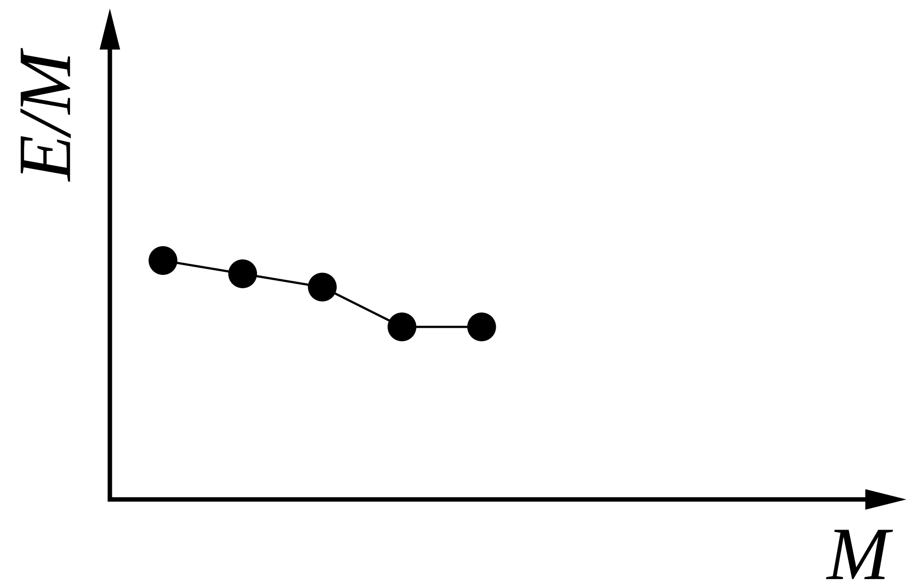

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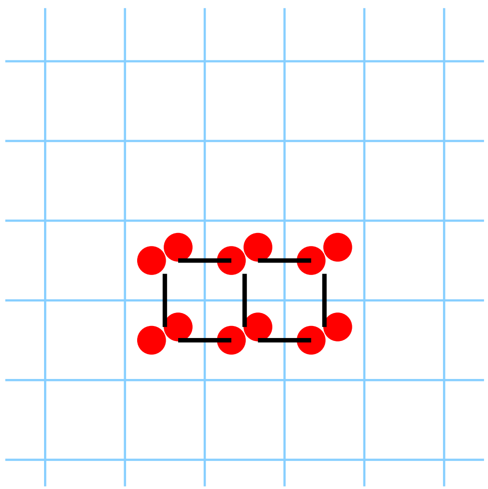

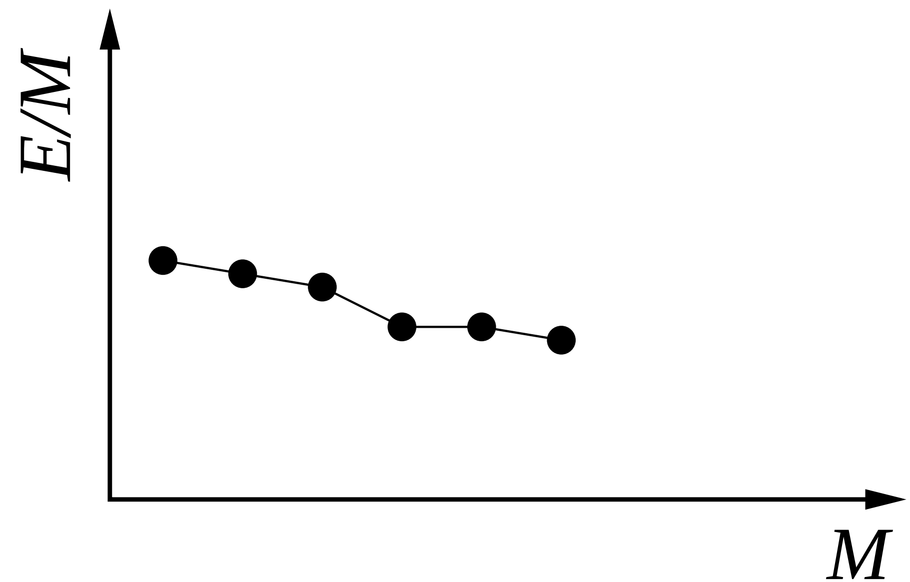

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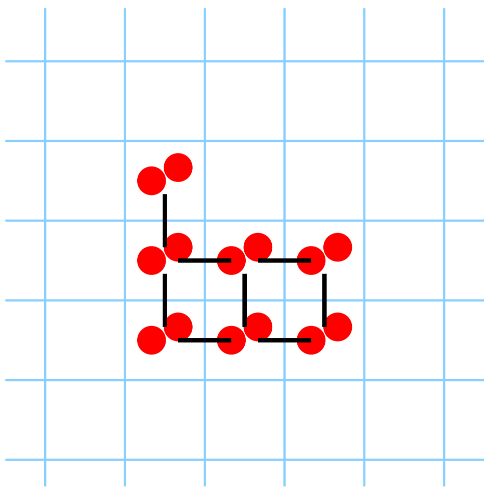

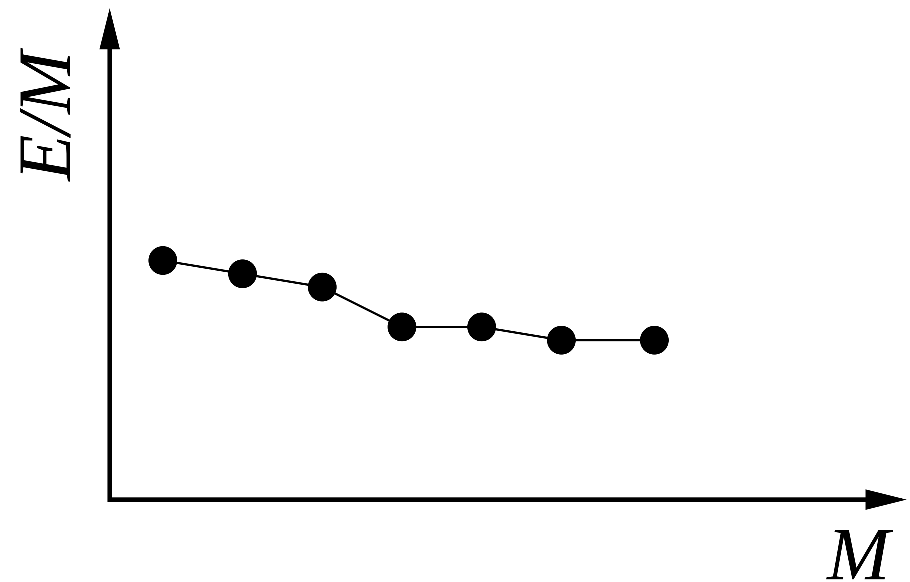

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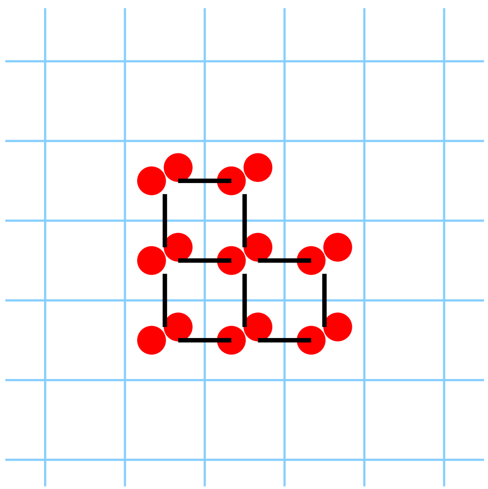

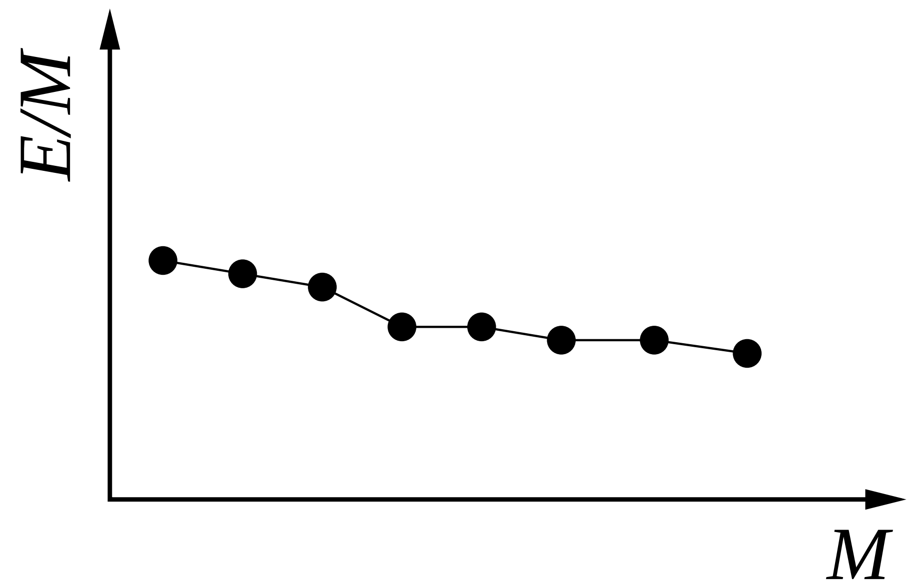

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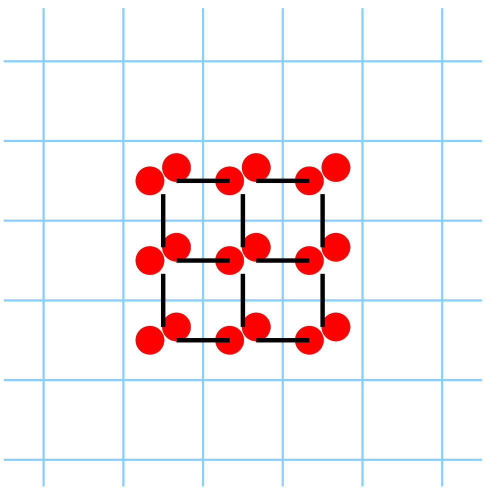

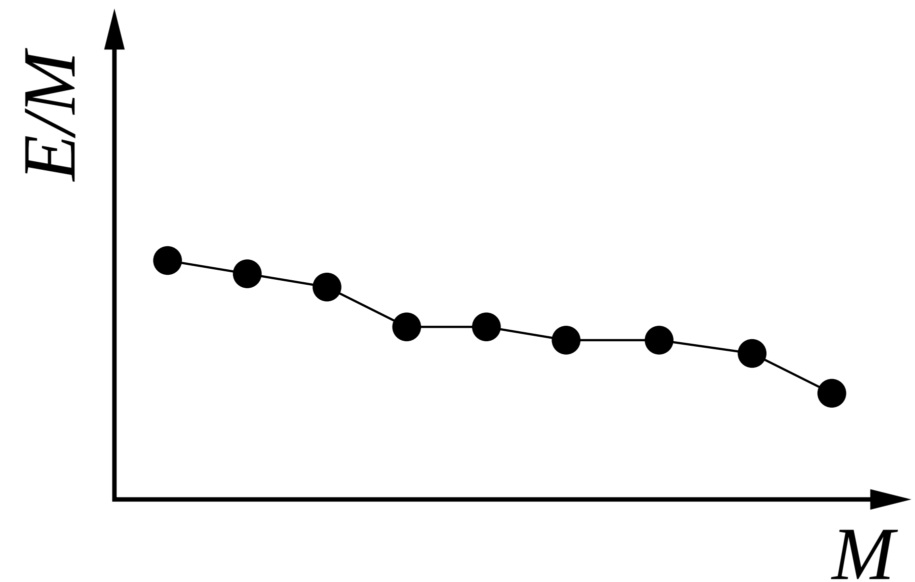

Spin-$\frac{1}{2}$ model ($|0\rangle \rightarrow |\downarrow\rangle$, $|1\rangle \rightarrow |\uparrow\rangle$) $\Rightarrow$ ferromagnetic spin domain

Attractively-bound dimers ($U < 0$)

Effective Hamiltonian ($m = 0, 1 \ \forall j$)

$$H_{\text{eff}}^{(0,1)} = \left[2\varepsilon + U - 2d\tilde{J}\right] \sum_j \hat{m}_j - \tilde{J} \sum_{\langle j,i \rangle} c_j^\dagger c_i + 4\tilde{J} \sum_{\langle j,i \rangle} \hat{m}_j \hat{m}_i$$

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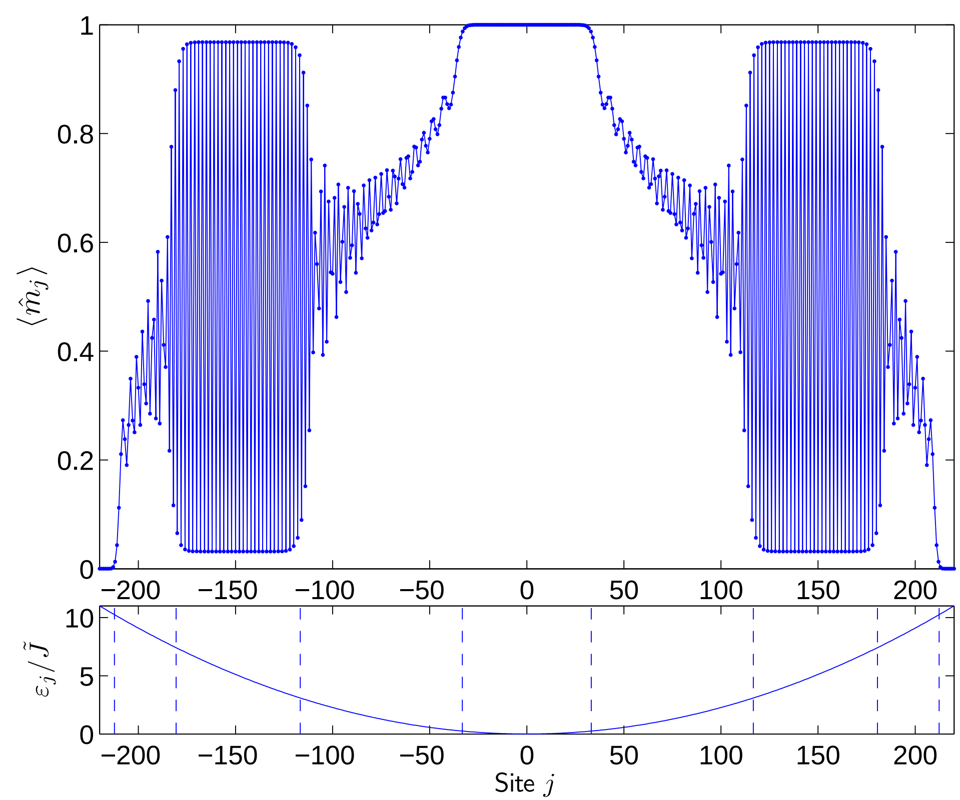
Extended Hubbard Model with $\tilde{J} > 0$ (nearest-neighbor repulsion)

Dimer density in 1D lattice + weak harmonic potential

$$2\varepsilon_j = \frac{j^2}{2200}\tilde{J}$$

Ground state from DMRG calculation



Schmidt, Bortz, Eggert, Fleischhauer, Petrosyan, Phys. Rev. A **79**, 063634 (2009)

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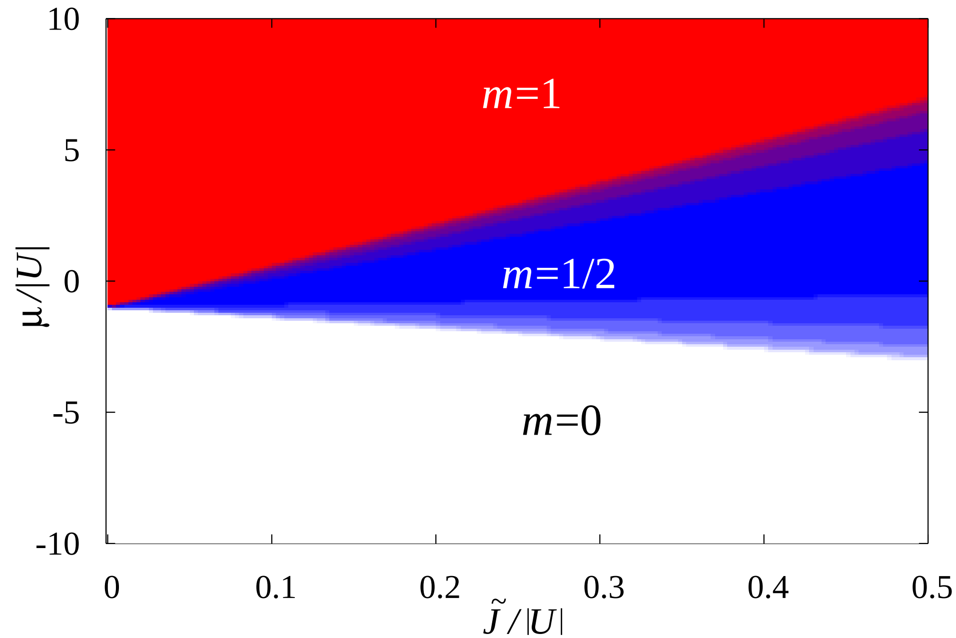
$H_{\rm eff}$ with uniform chem. potential

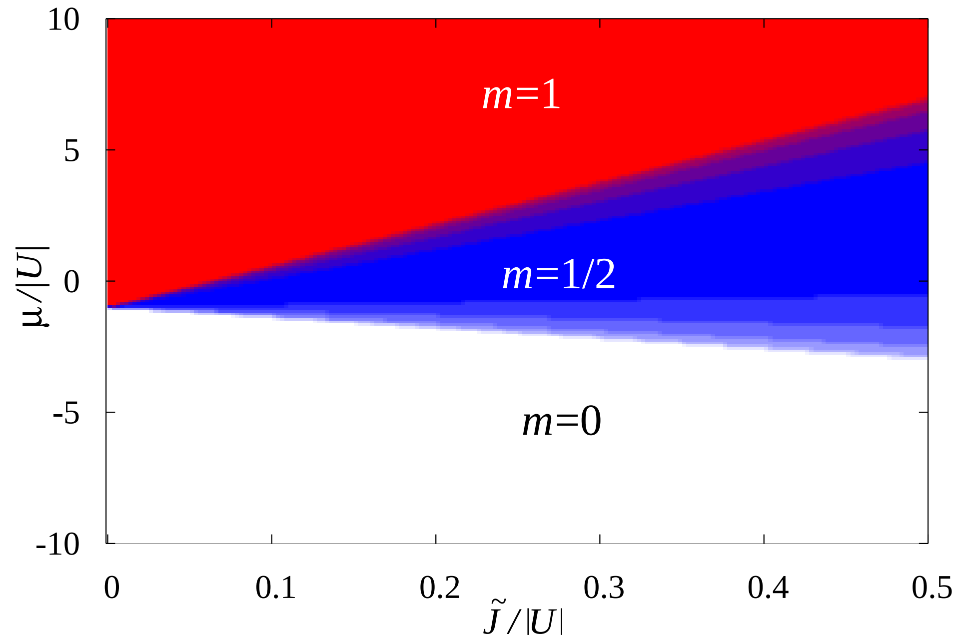
$$\boxed{\mu = -2\varepsilon}$$

Exacts diagonalization for 10 sites



Attractively-bound dimers ($U < 0$)

1D Phase Diagram $[\mu - \tilde{J}]$



Grand canonical ensemble

$H_{\rm eff}$ with uniform chem. potential

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$m = 0$, $m = \frac{1}{2}$, $m = 1$ incompressible phases:

- $m = 0 \rightarrow |0\rangle\,|0\rangle\,|0\rangle\,|0\rangle$ empty (ferromagnetic) phase
- $m = 1 \rightarrow |1\rangle\,|1\rangle\,|1\rangle\,|1\rangle$ filled (ferromagnetic) phase
- $m = \frac{1}{2} \rightarrow |0\rangle\,|1\rangle\,|0\rangle\,|1\rangle$ “crystal” (anti-ferromagnetic) phase

$0 < m < \frac{1}{2}$ & $\frac{1}{2} < m < 1$ compressible (supersolid) phases

Attractively-bound dimers ($U < 0$)

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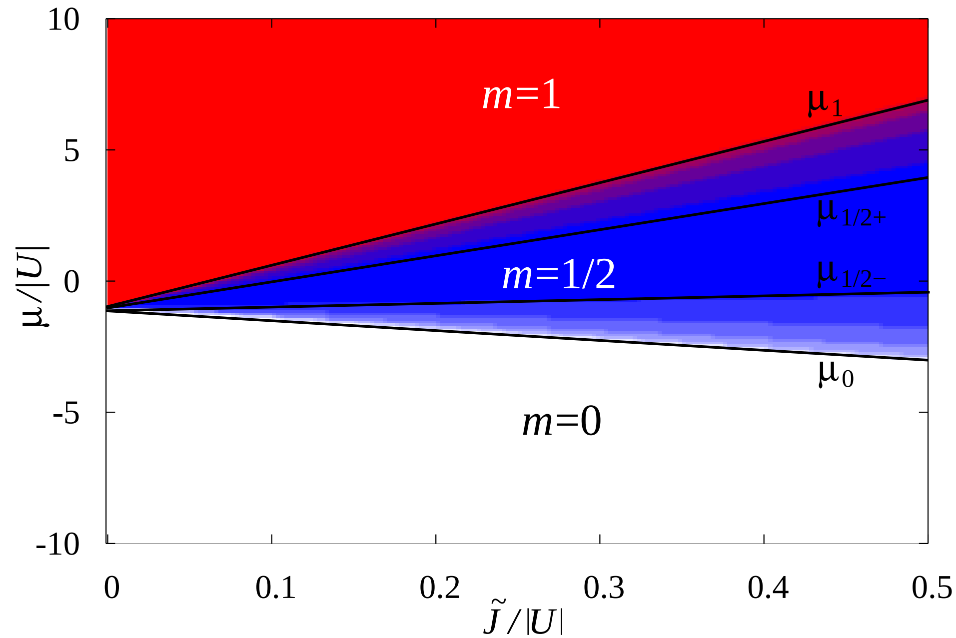

Exact Bethe Ansatz solution

$$\mu_0 = U - 4\tilde{J} \qquad \mu_1 = U + 16\tilde{J}$$

$$\mu_{1/2-} = U + 1.6836\tilde{J} \qquad \mu_{1/2+} = U + 10.3164\tilde{J}$$

Yang, Yang, Phys. Rev. **150**, 327 (1966)

Checkerboard crystal in a lattice ($U < 0$)

Strong dimer-dimer repulsion ($8\tilde{J} > \tilde{J}$)

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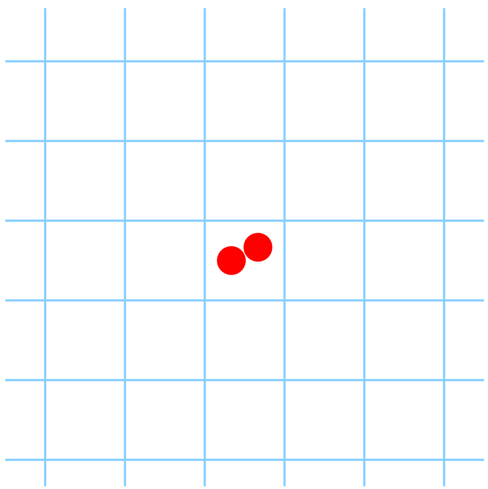

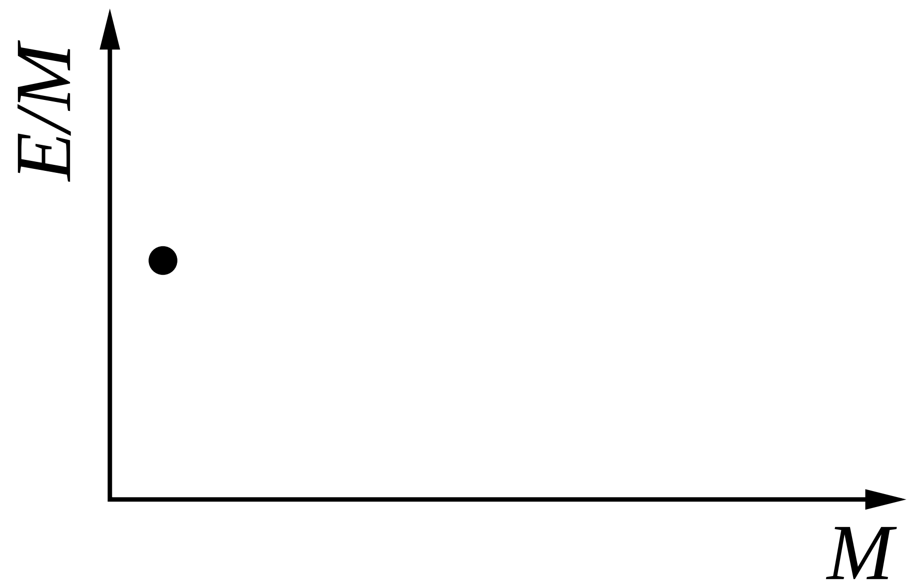

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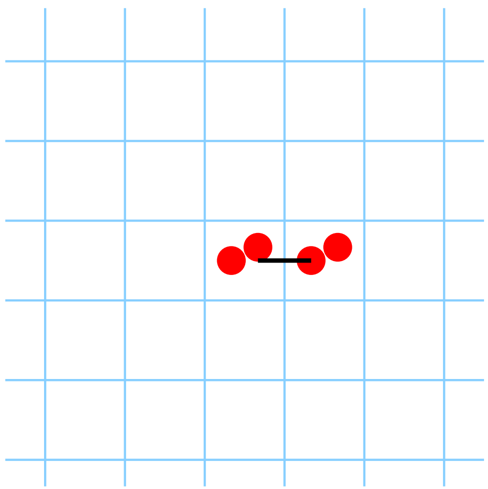

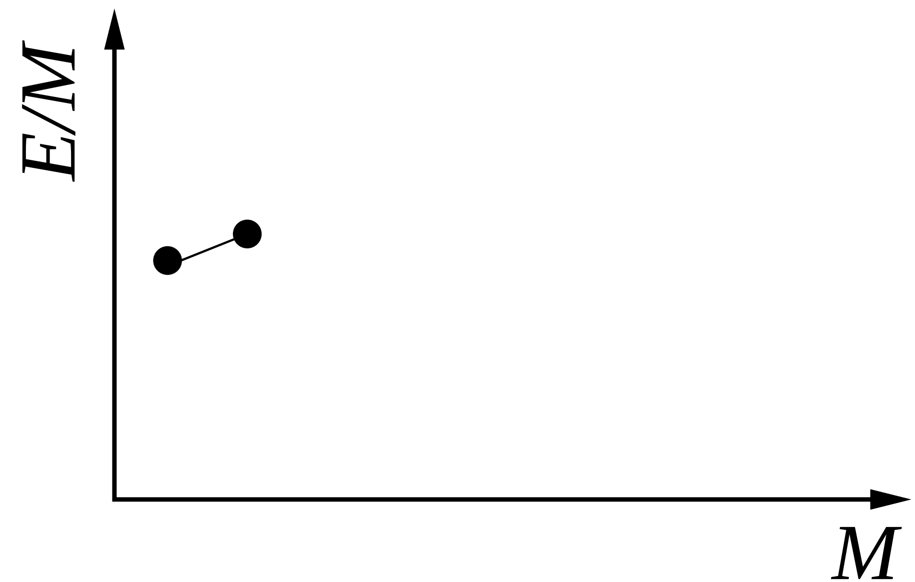

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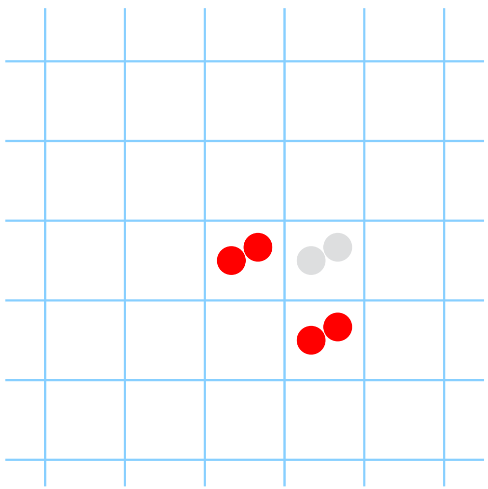

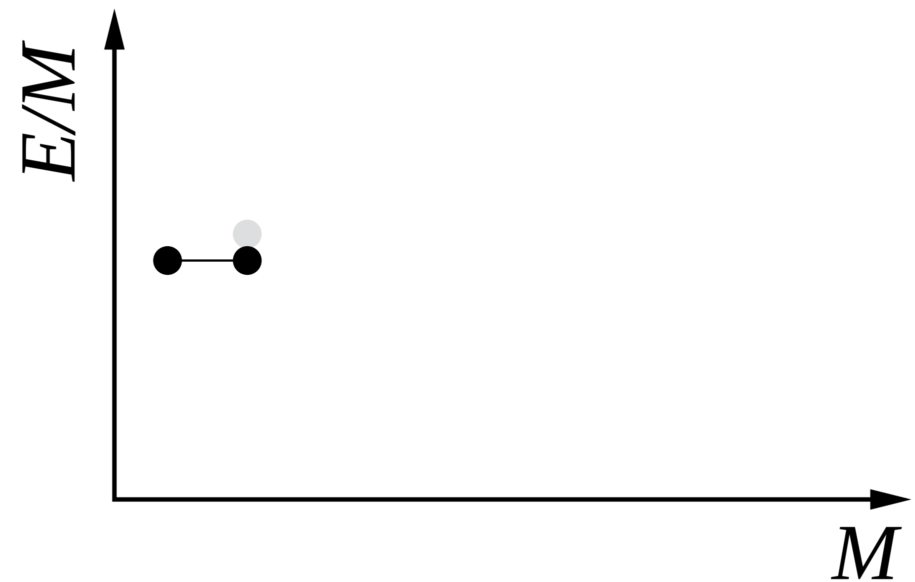

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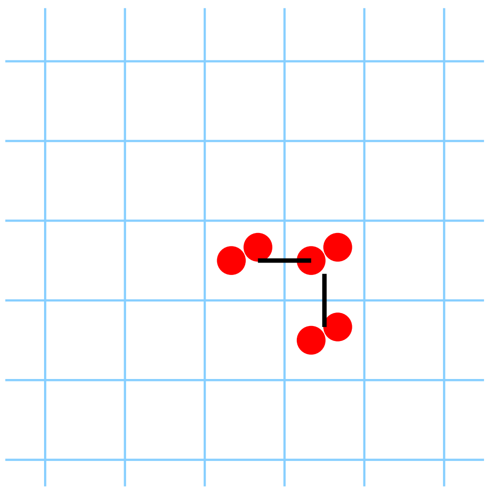

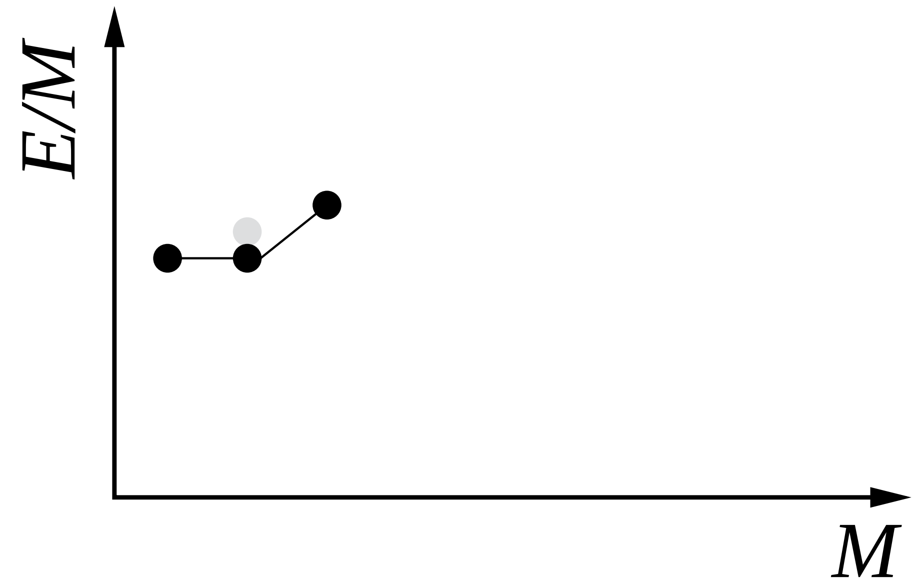

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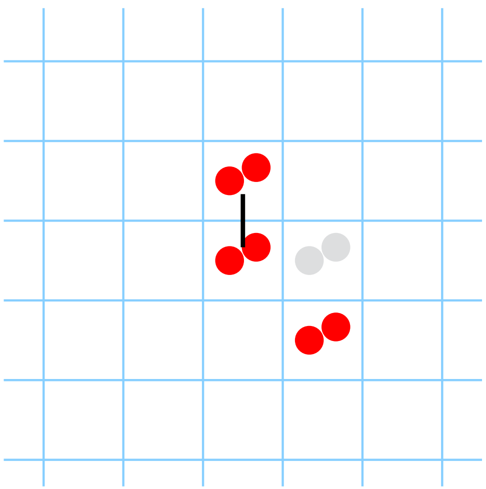

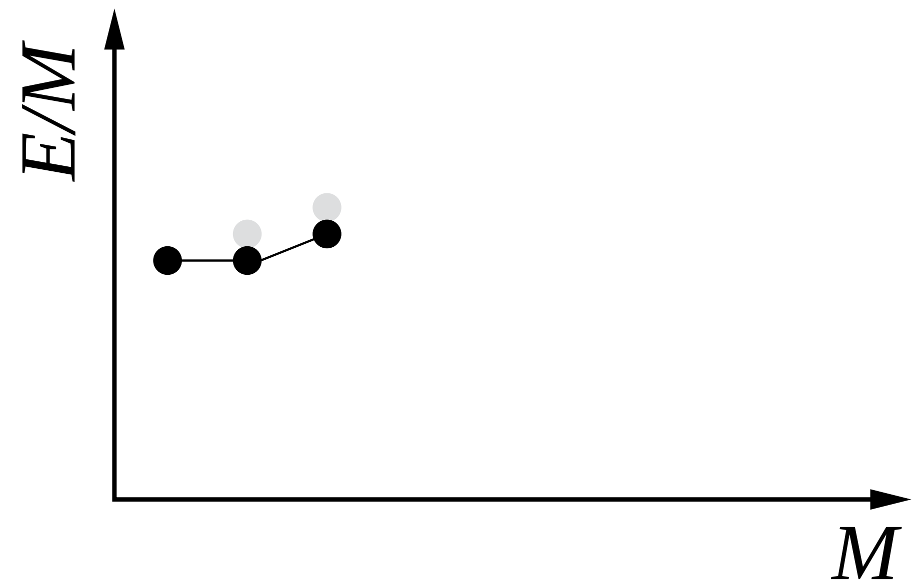

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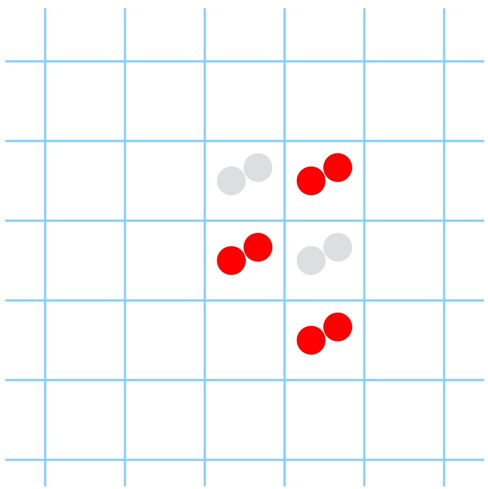

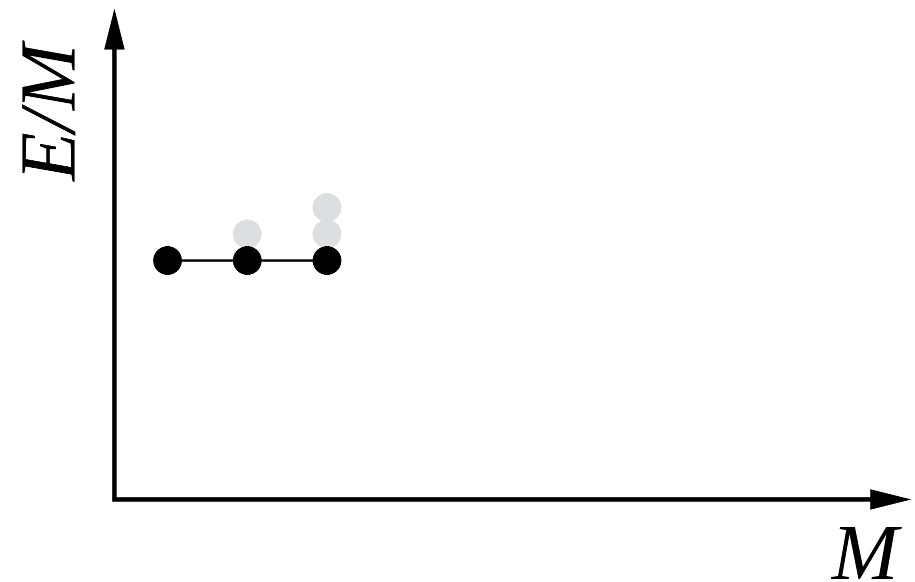

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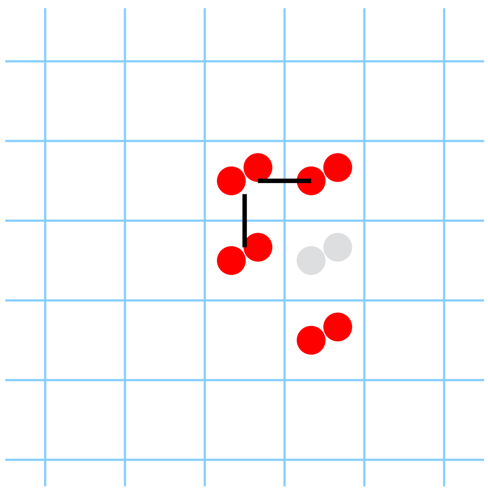

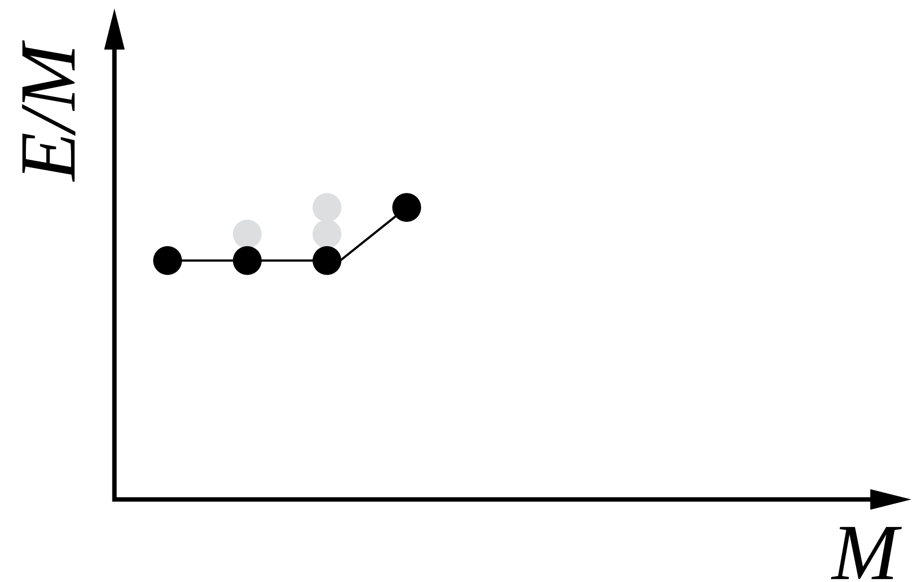

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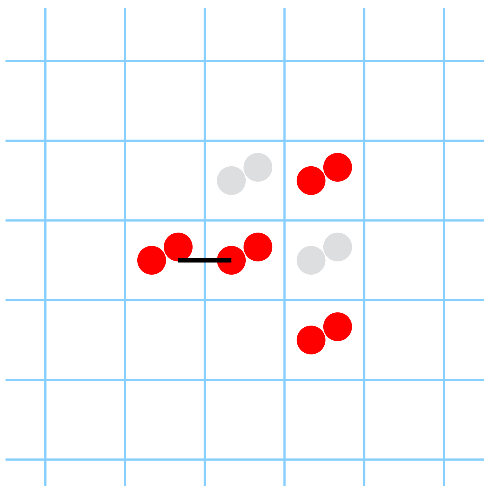

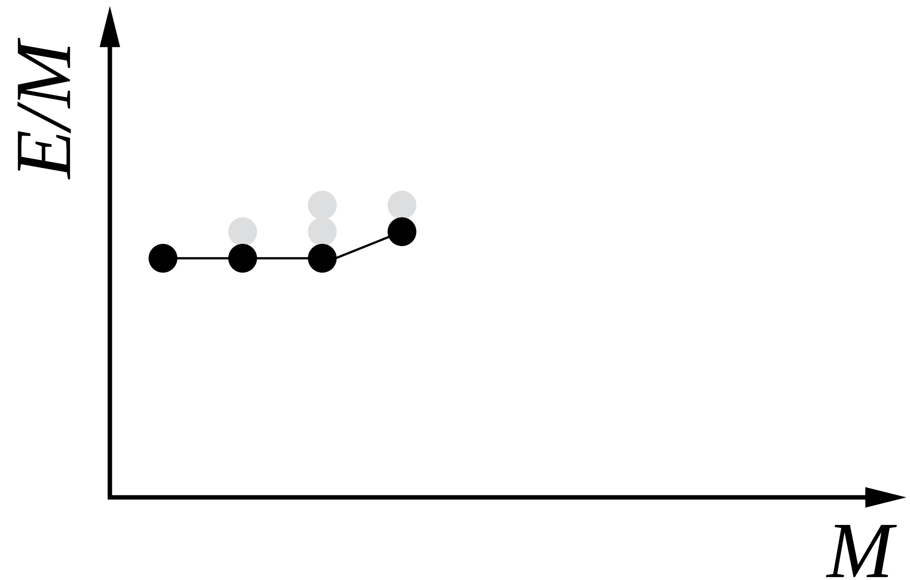

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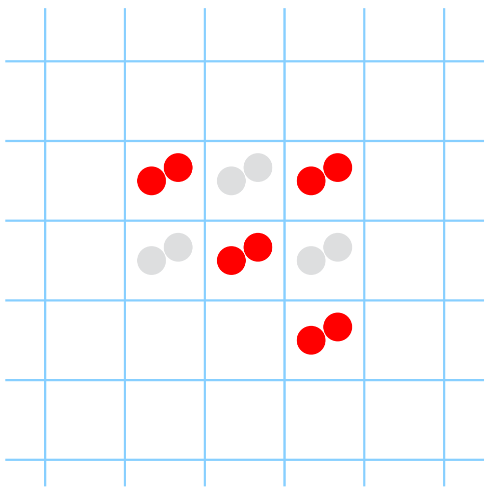

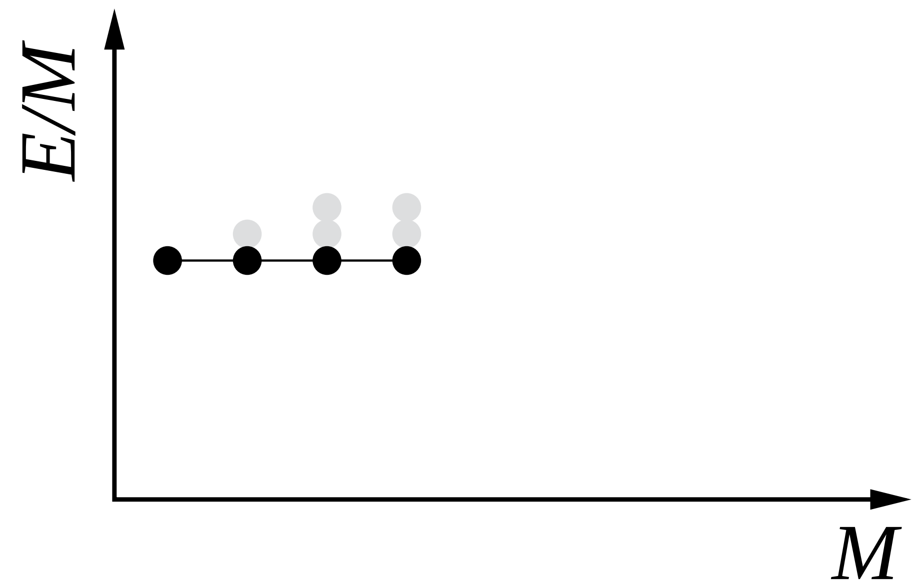

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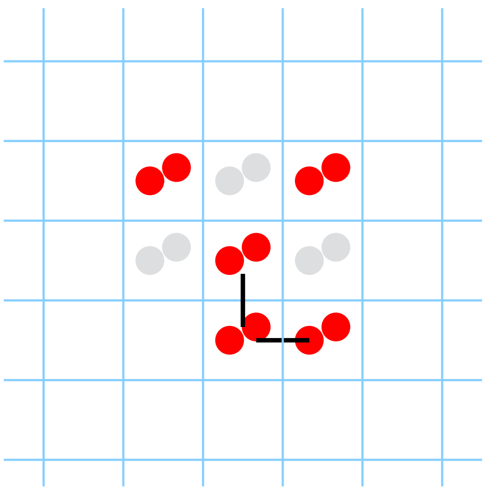

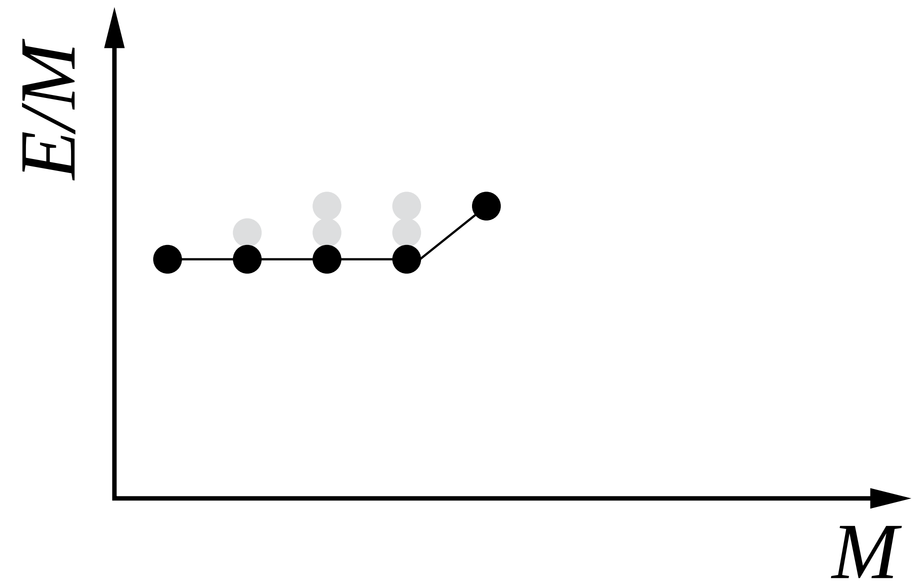

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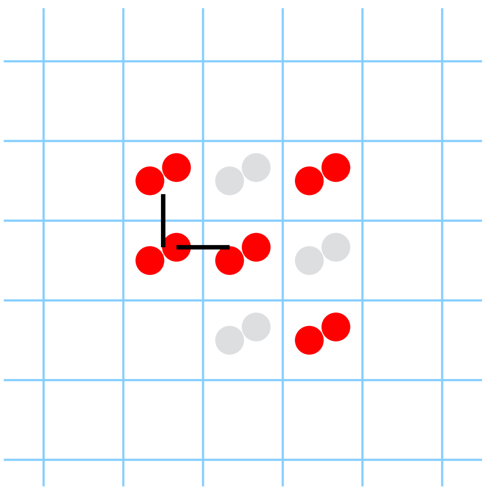

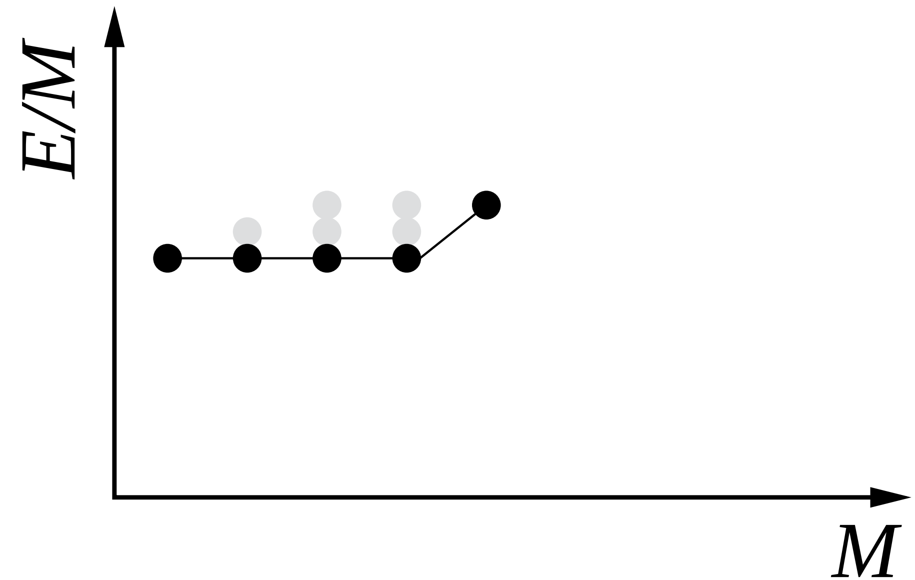

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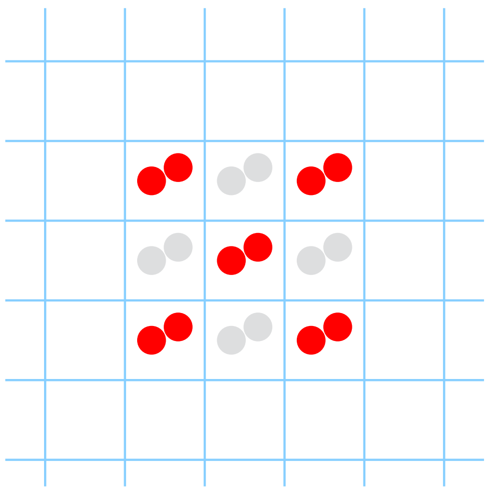

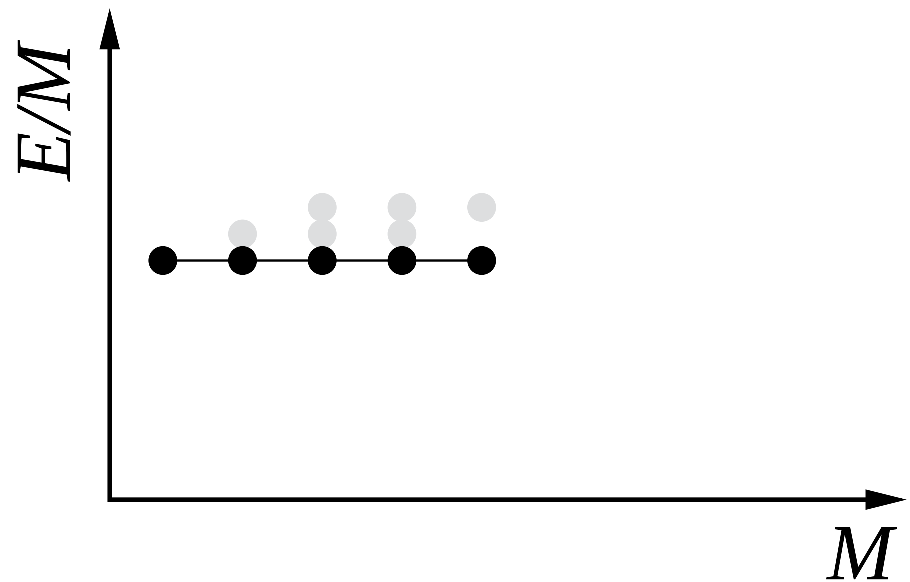

Spin-$\frac{1}{2}$ model ($|0\rangle \rightarrow |\downarrow\rangle$, $|1\rangle \rightarrow |\uparrow\rangle$) $\Rightarrow$ anti-ferromagnetic ordering

Summary

- Interaction (attraction or repulsion) can bind particles together in a lattice
- Strongly interacting pairs of particles form tightly-bound dimers
  - Dimer-monomer (particle) exchange interaction can bind them into trimers
- Collection of such dimers in a lattice can realize extended Hubbard (or spin-$\frac{1}{2}$ XXZ) model $\Rightarrow$ studies of many-body physics on a lattice

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Collaborators

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