# Basics of coherent light-matter interactions

**David Petrosyan**

# Outline

- Atom in an external radiation field
  - Dipole coupling and the selection rules
  - Spontaneous decay of an excited atom

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  - Rabi oscillations
  - Eigenstates
  - Adiabatic and non-adiabatic transitions

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  - Eigenstates
  - Adiabatic and non-adiabatic transitions
- Three-level atom in a (bichromatic) laser field
  - Two-photon Rabi oscillations
  - Eigenstates: Coherent population trapping (dark) state
  - Stimulated Raman adiabatic passage (STIRAP)

# I. Atom in an external radiation field

# Atomic Level Structure

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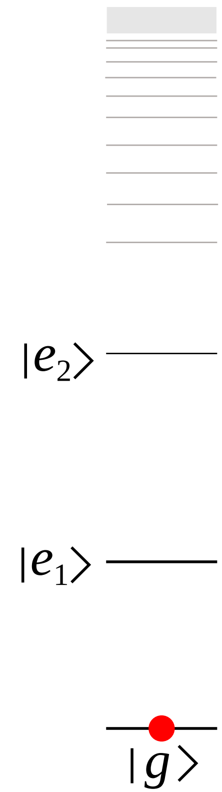
effective PQN $n^* = n - \delta_l$ ($\delta_l$ quantum defect)



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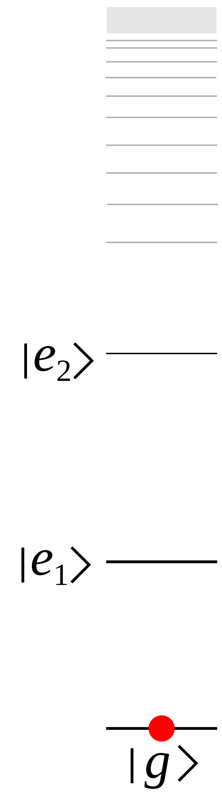
Wavefunctions $\boxed{\langle \boldsymbol{r} | nlm \rangle \equiv \Psi_{nlm}(\boldsymbol{r}) = R_{nl}(r) Y_l^m(\theta, \phi)}$

OAMQN $l = 0, 1, \ldots, n-1$ MQN (projection) $m = -l, -l+1, \ldots, l$

[e.g. $R_{10}(r) = 2a_0^{-3/2} e^{-r/a_0}$ $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}$]



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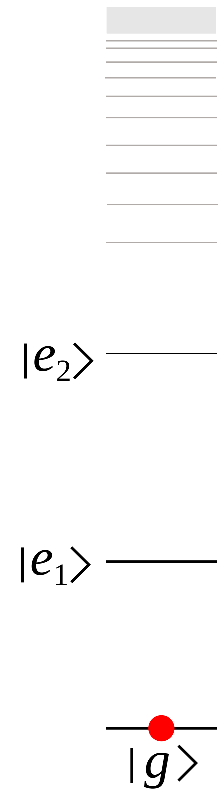


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Parity of $|nlm\rangle$:

- $l = 0, 2, 4, \ldots$ (s,d,g,. . . ) $\Rightarrow \Psi_{nlm}(\boldsymbol{r}) = \Psi_{nlm}(-\boldsymbol{r})$ even
- $l = 1, 3, 5, \ldots$ (p,f,h,. . . ) $\Rightarrow \Psi_{nlm}(\boldsymbol{r}) = -\Psi_{nlm}(-\boldsymbol{r})$ odd

# Spin-Orbit Coupling

**Fine structure**: electron spin $\boldsymbol{s}$ interacts with orbital angular momentum $\boldsymbol{l}$

$\Rightarrow$ *LS* interaction $\mathcal{V}_{ls} \propto \boldsymbol{l} \cdot \boldsymbol{s} \sim \alpha^2 E_n$

$\Rightarrow$ Total angular momentum $\boldsymbol{J} = \boldsymbol{l} + \boldsymbol{s}$

$$|J, M\rangle = \sum_{\substack{m_l, m_s \\ (m_l + m_s = M)}} C^j_{m_l m_s} \, |l m_l\rangle \, |s m_s\rangle$$

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**Hyperfine structure**: nuclear spin $\boldsymbol{I}$ interacts with $\boldsymbol{l}$ and $\boldsymbol{s}$

$\Rightarrow$ *LI* interaction $V_{LI} \propto \boldsymbol{l} \cdot \boldsymbol{I} \sim \frac{m_e}{m_p} \alpha^2 E_n$

$\Rightarrow$ Total angular momentum $\boldsymbol{F} = \boldsymbol{J} + \boldsymbol{I}$

# Atom-Field Coupling

$$\mathcal{H} = \frac{1}{2m_\mathrm{e}}[\boldsymbol{P} - e\boldsymbol{A}(\boldsymbol{r})]^2 + V(r) = \mathcal{H}^\mathrm{A} + \mathcal{V}^\mathrm{AF}$$

$$\Rightarrow \quad \mathcal{V}^\mathrm{AF} = -\frac{e}{m_\mathrm{e}}\boldsymbol{P}\cdot\boldsymbol{A} + \frac{e^2}{2m_\mathrm{e}}\boldsymbol{A}^2$$

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**Dipole approximation** $\quad \boldsymbol{k}\cdot\boldsymbol{r} \lesssim ka_0 \ll 1 \quad \Rightarrow \quad \boldsymbol{A}(\boldsymbol{r}) \simeq \boldsymbol{A}(0)$

$$\langle nlm_l|\,\mathcal{V}^\mathrm{AF}\,|n'l'm_l'\rangle = -\frac{e}{m_\mathsf{e}}\langle nlm_l|\,\boldsymbol{P}\,|n'l'm_l'\rangle\cdot\boldsymbol{A}$$

$$= i\frac{e}{\hbar}\langle nlm_l|\,[\boldsymbol{r},\mathcal{H}^\mathrm{A}]\,|n'l'm_l'\rangle\cdot\boldsymbol{A}$$

$$= -i\omega_{nn'}e\langle nlm_l|\,\boldsymbol{r}\,|n'l'm_l'\rangle\cdot\boldsymbol{A} \qquad \left[\omega_{nn'} = \frac{E_n - E_{n'}}{\hbar}\right]$$

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For $\omega \sim \omega_{nn'} \quad \Rightarrow \quad i\omega\boldsymbol{A} = \boldsymbol{E} \quad \Rightarrow$

**Dipole coupling** $\quad \boxed{\mathcal{V}^\text{AF} = -e\boldsymbol{r}\cdot\boldsymbol{E}}$

# Selection Rules for Dipole Transitions

$$\langle nlm|\, e\boldsymbol{r}\, |n'l'm'\rangle \equiv e\int d^3r\, \Psi^*_{nlm}(\boldsymbol{r})\, \boldsymbol{r}\, \Psi_{n'l'm'}(\boldsymbol{r}) \neq 0 \quad \text{for} \quad l - l' = \pm 1$$

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**Linearly $\pi$ polarized field** $(\hat{z} \parallel \boldsymbol{E})$

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$\langle JM|\, e\boldsymbol{r}\,|J'M'\rangle \neq 0$ for $J - J' = 0, \pm 1$ & $M - M' = m_{\text{phot}}$

($F \leftrightarrow J$)

# Spontaneous Decay of an Excited Atom

**Free-space EM field**: $\boldsymbol{E} = -i \sum_{\boldsymbol{k}\sigma} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \hat{\boldsymbol{e}}_{\boldsymbol{k}\sigma} [a_{\boldsymbol{k}\sigma} - a^\dagger_{\boldsymbol{k}\sigma}]; \quad \mathcal{V}^{\mathrm{AF}} = -e\boldsymbol{r} \cdot \boldsymbol{E}$

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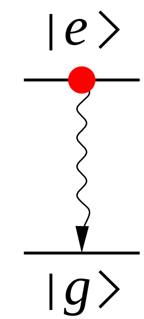
$$|\Psi(t)\rangle = c_e(t)\,|e,0\rangle + \sum_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}(t)\,|g, 1_{\boldsymbol{k}\sigma}\rangle \quad \Rightarrow$$



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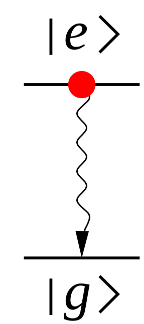


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$$\frac{\partial}{\partial t} c_{\boldsymbol{k}\sigma} = i(\omega_{eg} - \omega_k) c_{\boldsymbol{k}\sigma} - \frac{i}{\hbar} \langle g, 1_{\boldsymbol{k}\sigma}|\, \mathcal{V}^{\mathrm{AF}}\, |e, 0\rangle\, c_e$$

$$c_{\boldsymbol{k}\sigma}(t) = -\frac{i}{\hbar} \int_0^t e^{i(\omega_{eg} - \omega_k)(t - t')} \langle g, 1_{\boldsymbol{k}\sigma}|\, \mathcal{V}^{\mathrm{AF}}\, |e, 0\rangle\, c_e(t')\, dt'$$

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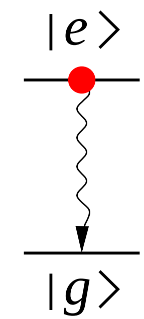


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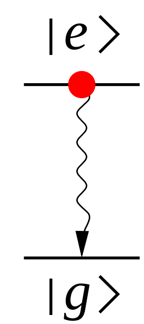


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&= -\frac{1}{\hbar^2}\sum_{\boldsymbol{k}\sigma}|\langle g, 1_{\boldsymbol{k}\sigma}|\,\mathcal{V}^{\mathrm{AF}}\,|e,0\rangle|^2 \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')} c_e(t')\,dt' \\
&\simeq -\frac{1}{\hbar^2}\sum_{\boldsymbol{k}\sigma}|\langle g, 1_{\boldsymbol{k}\sigma}|\,\mathcal{V}^{\mathrm{AF}}\,|e,0\rangle|^2 \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')} dt'\, c_e(t) \\
&\equiv -G_e c_e(t)
\end{aligned}$$

# Decay Rate and Level Shift

$$G_e \equiv \sum_{\boldsymbol{k}\sigma} \frac{\omega_k}{2\varepsilon_0 \hbar V} |\boldsymbol{\wp}_{ge} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{k}\sigma}|^2 \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')} dt' \qquad (\boldsymbol{\wp}_{ge} = \langle g| e\boldsymbol{r} |e\rangle)$$

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$$\Gamma_e = \frac{1}{4\pi\varepsilon_0} \frac{4\omega_{eg}^3 |\wp_{eg}|^2}{3\hbar c^3} \qquad S_e = \frac{1}{4\pi\varepsilon_0} \frac{2|\wp_{eg}|^2}{3\pi\hbar c^3} \mathrm{P} \int d\omega_k \frac{\omega_k^3}{\omega_{eg}-\omega_k}$$

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$$\boxed{\frac{\partial}{\partial t}|c_e|^2 = -\Gamma_e |c_e|^2} \quad \Rightarrow \quad |c_e(t)|^2 = e^{-\Gamma_e t}|c_e(0)|^2$$

# Problems & Questions

- Prove the commutation relation $[\boldsymbol{r}, \mathcal{H}^{\mathrm{A}}] = i\frac{\hbar}{m_{\mathrm{e}}}\boldsymbol{P}$
- $\Gamma_{eg}$ is the spontaneous decay rate of an excited atomic state $|e\rangle$ to the ground state $|g\rangle$ by emitting a photon into the vacuum modes of the radiation field.
  If the initial state of the field was not a vacuum $n = 0$, but, e.g., a thermal state [with Plank distribution $\bar{n}(\omega_k) = (e^{\hbar\omega_k/k_{\mathrm{B}}T} - 1)^{-1}$], what would the decay rate $\Gamma_{eg}$ be?
  Could an atom in the ground state $|g\rangle$ be excited to $|e\rangle$ by the thermal photons? What would the excitation rate $\Gamma_{ge}$ be?

# II. Two-level atom in a laser field

# Atom-Field Interaction

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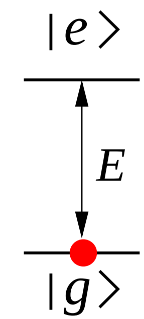

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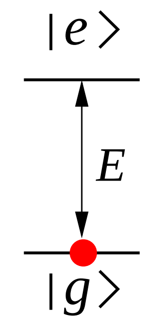


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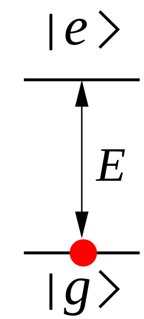


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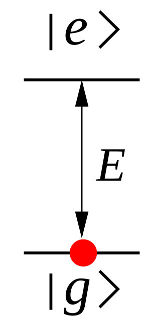
$$\mathcal{H}^{\mathrm{A}} = \hbar\omega_g \, |g\rangle\langle g| \, + \hbar\omega_e \, |e\rangle\langle e|$$

$$\mathcal{V}^{\mathrm{AF}}(t) = -\boldsymbol{\wp} \cdot \boldsymbol{E}(t) = -\wp E(t)$$ with

$$\wp \equiv \boldsymbol{\wp} \cdot \boldsymbol{e} \qquad E(t) = \mathcal{E} e^{-i\omega t} + \mathcal{E}^* e^{i\omega t} = 2|\mathcal{E}| \cos(\omega t - \varphi)$$

# Atom-Field Interaction




$$\mathcal{H} = \mathcal{H}^{\rm A} + \mathcal{V}^{\rm AF}(t)$$

$$\mathcal{H}^{\rm A} = \hbar\omega_g \,|g\rangle\langle g|\, + \hbar\omega_e \,|e\rangle\langle e|$$

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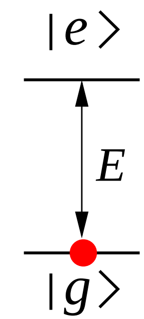


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**Interaction picture** $c_{g,e}(t) = \tilde{c}_{g,e}(t)e^{-i\omega_{g,e}t}$

$$\frac{\partial}{\partial t}\tilde{c}_g = i\tilde{c}_e\frac{\wp_{ge}}{\hbar}\left(\mathcal{E}e^{-i(\omega+\omega_{eg})t} + \mathcal{E}^* e^{i(\omega-\omega_{eg})t}\right) \simeq i\tilde{c}_e\frac{\wp_{ge}}{\hbar}\mathcal{E}^* e^{i\Delta t}$$

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Rotating Wave Approximation (RWA): $\omega + \omega_{eg} \gg \omega - \omega_{eg} \equiv \Delta$

# Rabi Oscillations

$$\frac{\partial}{\partial t} c_g = i\Omega^* c_e$$

$$\boxed{\Omega \equiv \frac{\wp_{eg}}{\hbar}\mathcal{E}}$$ Rabi frequency

$$\frac{\partial}{\partial t} c_e = i\Delta c_e + i\Omega c_g \qquad (c_e = \tilde{c}_e e^{i\Delta t})$$

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**Solution** $c_g(0) = 1, \quad c_e(0) = 0$

$$c_g(t) = \cos(\bar{\Omega}t) - i\frac{\Delta}{2\bar{\Omega}}\sin(\bar{\Omega}t)$$

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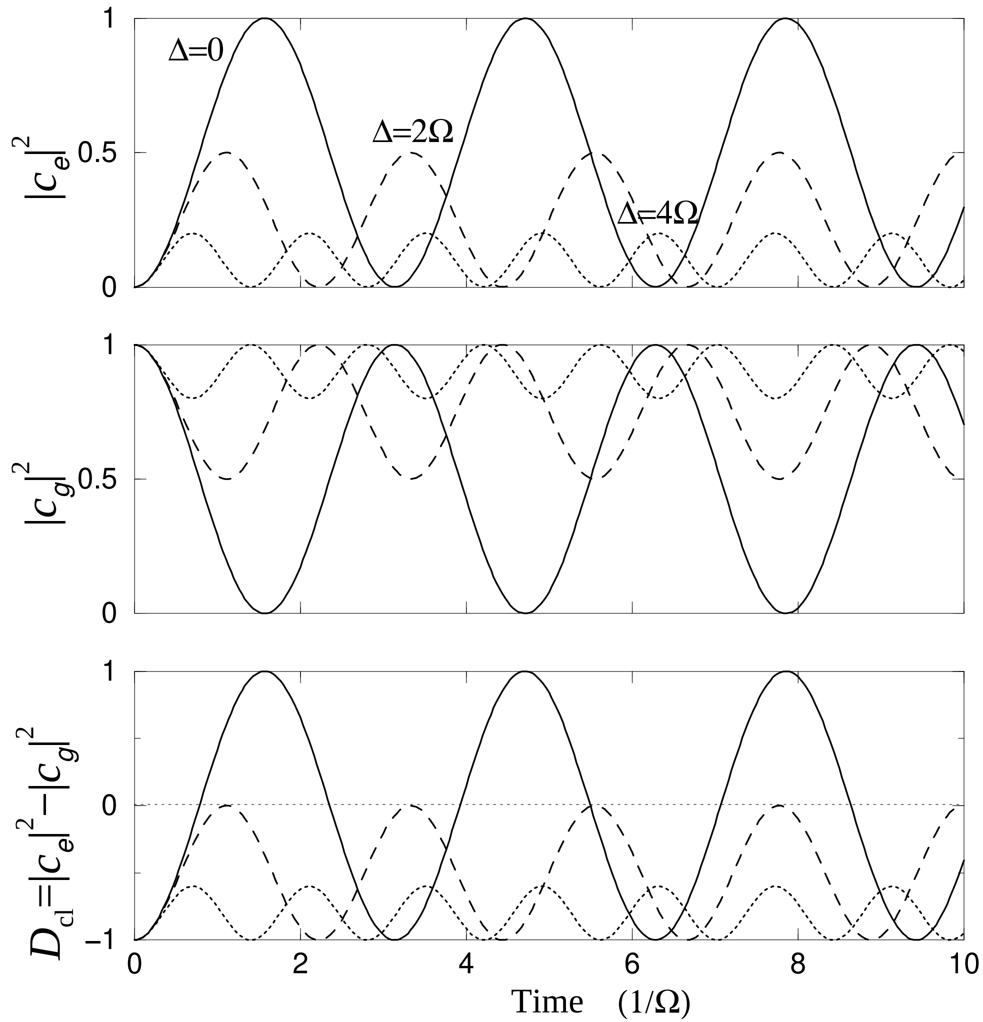

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**Resonant field** $\Delta = 0 \;\&\; \bar{\Omega} = \Omega$

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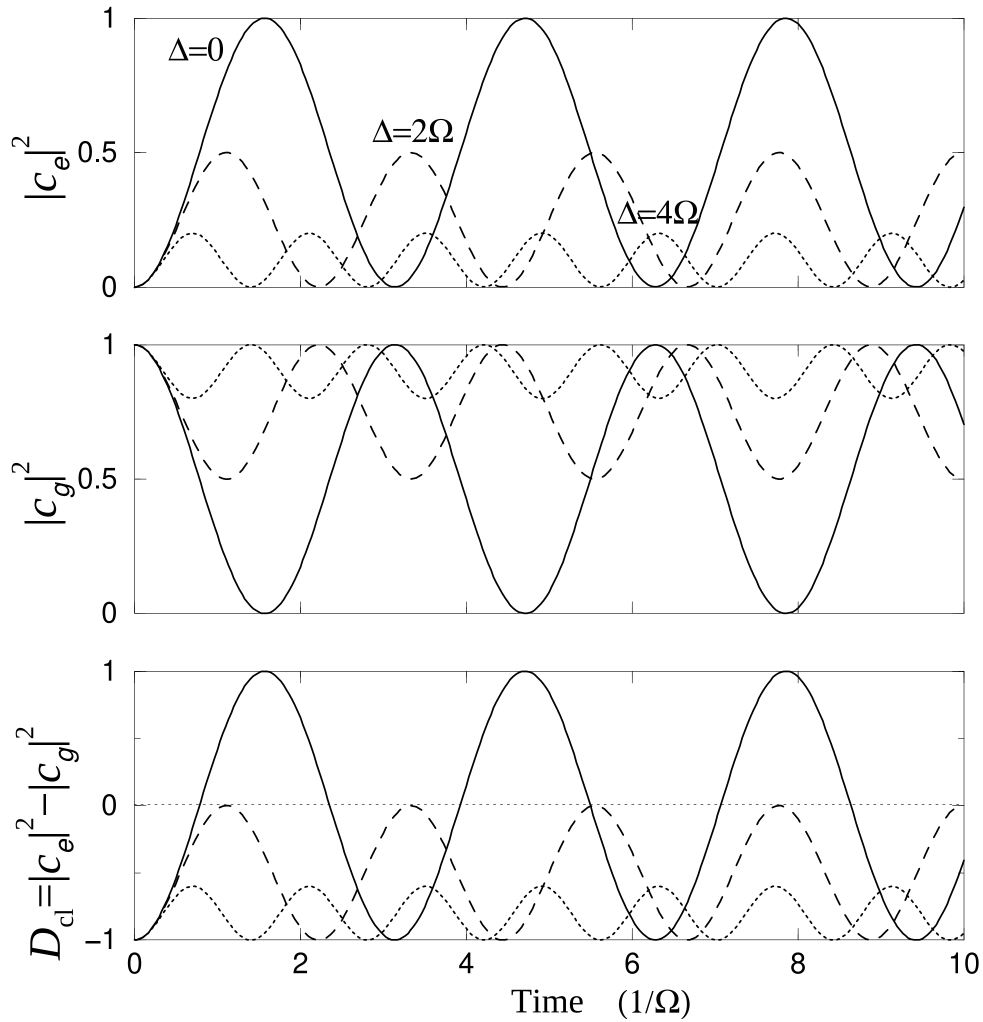

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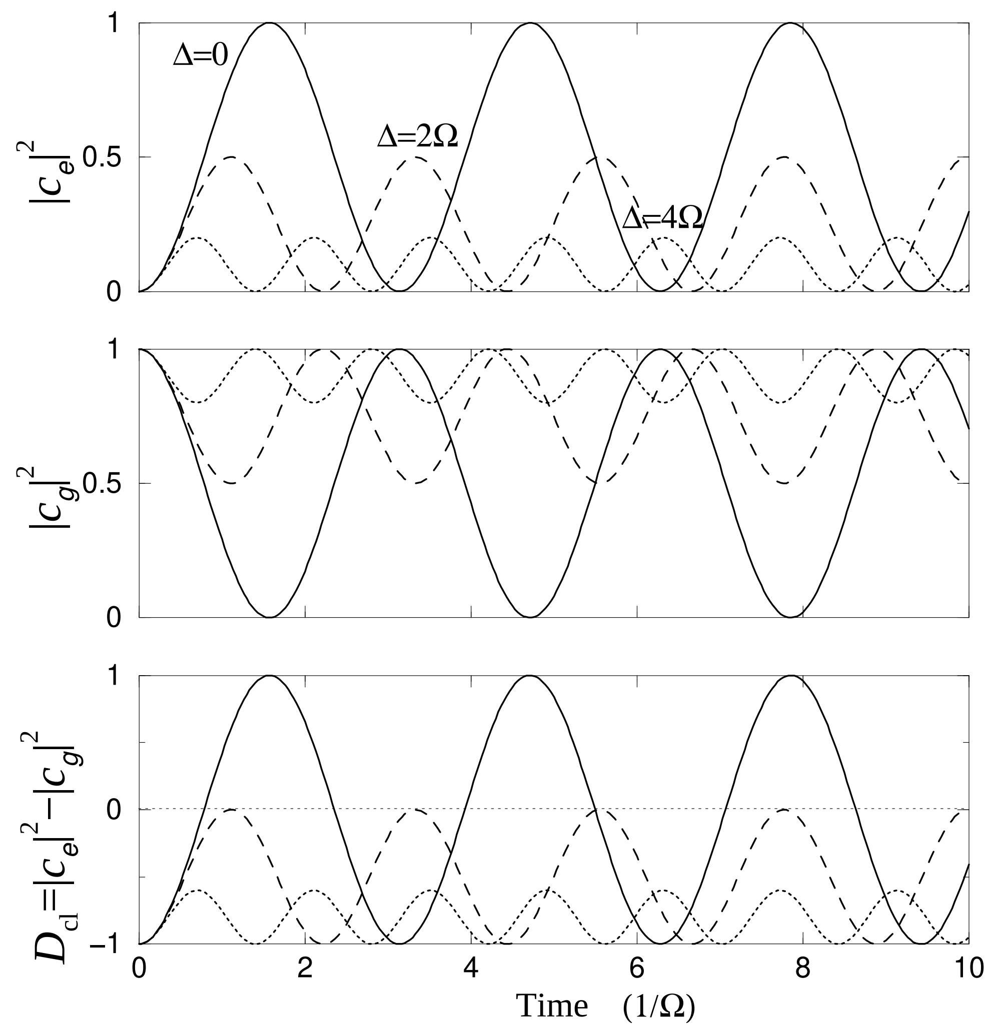
**Inversion** $2\Omega T = \pi$-pulse

$$|c_g(T)|^2 = 0 \quad |c_e(T)|^2 = 1$$



# Damped Rabi Oscillations

$$\frac{\partial}{\partial t}\rho_{gg} = \Gamma\rho_{ee} + i(\Omega^*\rho_{eg} - \rho_{ge}\Omega)$$

$$\frac{\partial}{\partial t}\rho_{ee} = -\Gamma\rho_{ee} + i(\Omega\rho_{ge} - \rho_{eg}\Omega^*)$$

$$\frac{\partial}{\partial t}\rho_{eg} = (i\Delta - \gamma_{eg})\rho_{eg} - i\Omega(\rho_{ee} - \rho_{gg})$$

$$\gamma_{eg} = \frac{1}{2}\Gamma + \ldots$$

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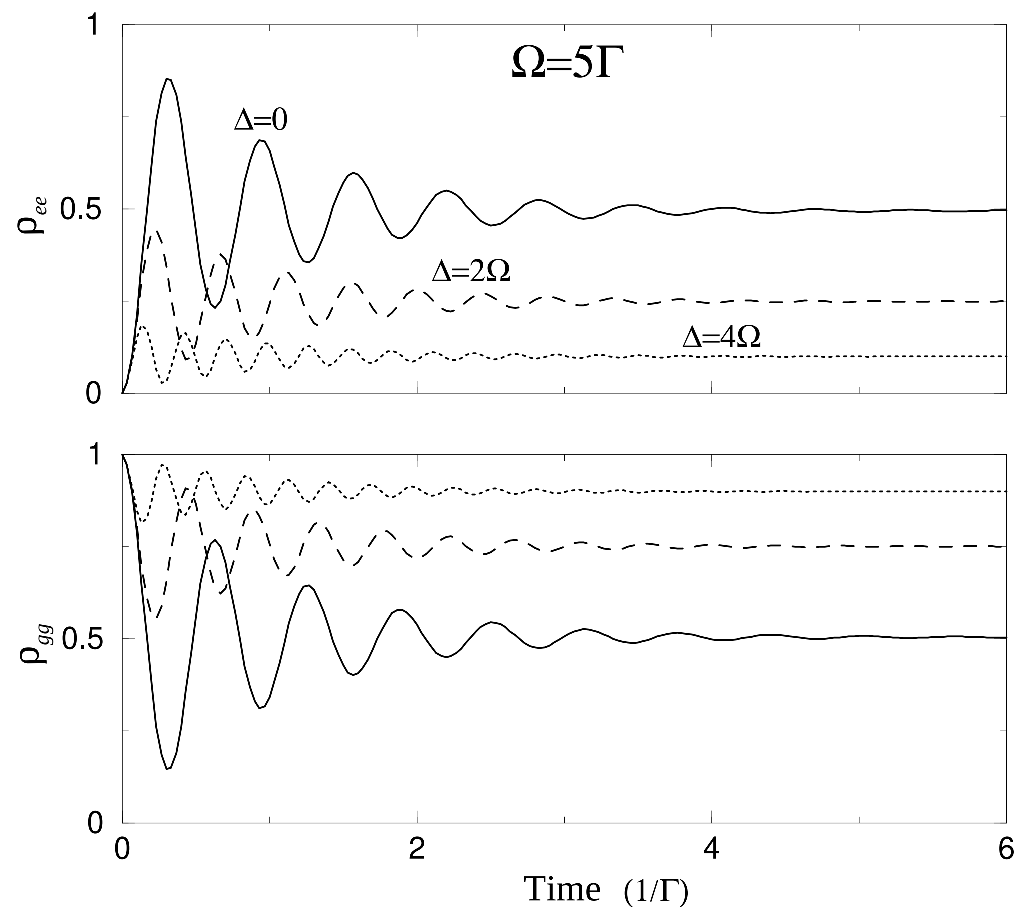
## Numerical solution



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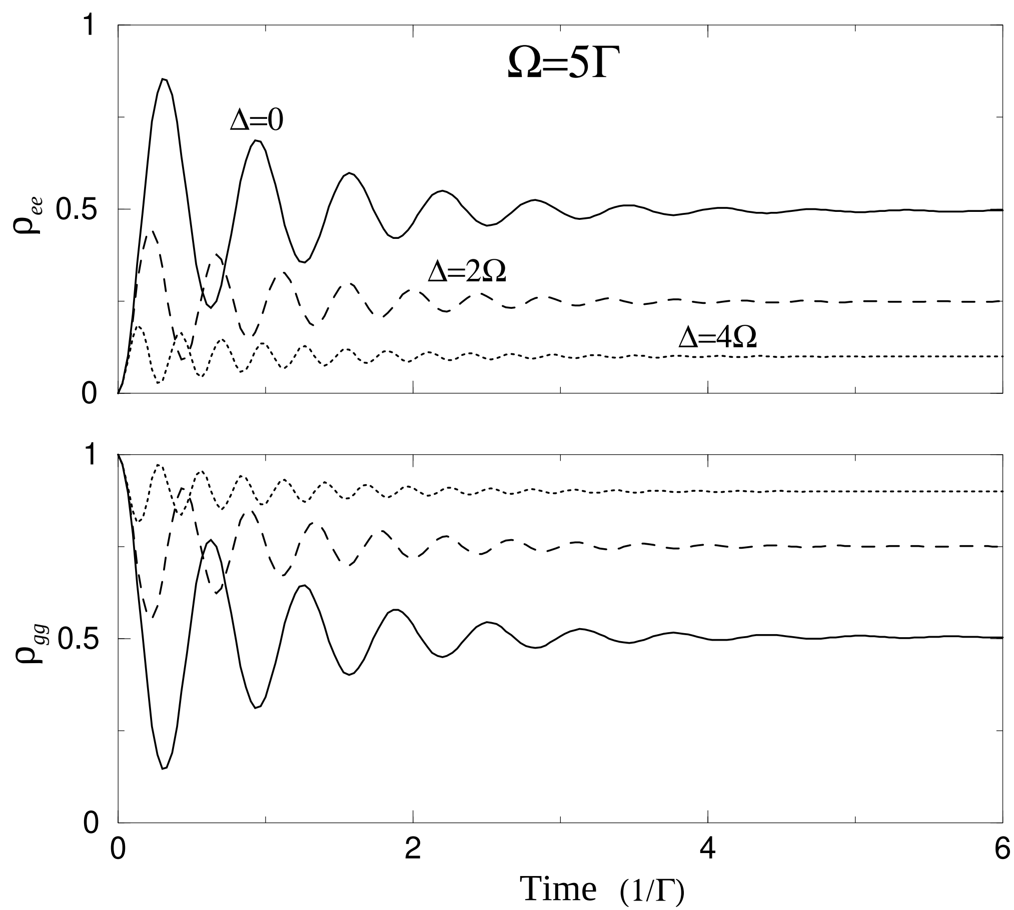
**Steady state** $t \to \infty$

$$\rho_{ee}(\infty) = \frac{|\Omega|^2}{\frac{\Gamma}{2\gamma_{eg}}(\Delta^2+\gamma_{eg}^2)+2|\Omega|^2}$$

$$\rho_{gg} = 1 - \rho_{ee}$$

$$w = \sqrt{2|\Omega|^2\frac{2\gamma_{eg}}{\Gamma} + \gamma_{eg}^2}$$ linewidth

**Numerical solution**



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$\rho_{ee}(\infty) \to \frac{1}{2}$ for $\Omega^2 \gg \Gamma\gamma_{eg}$

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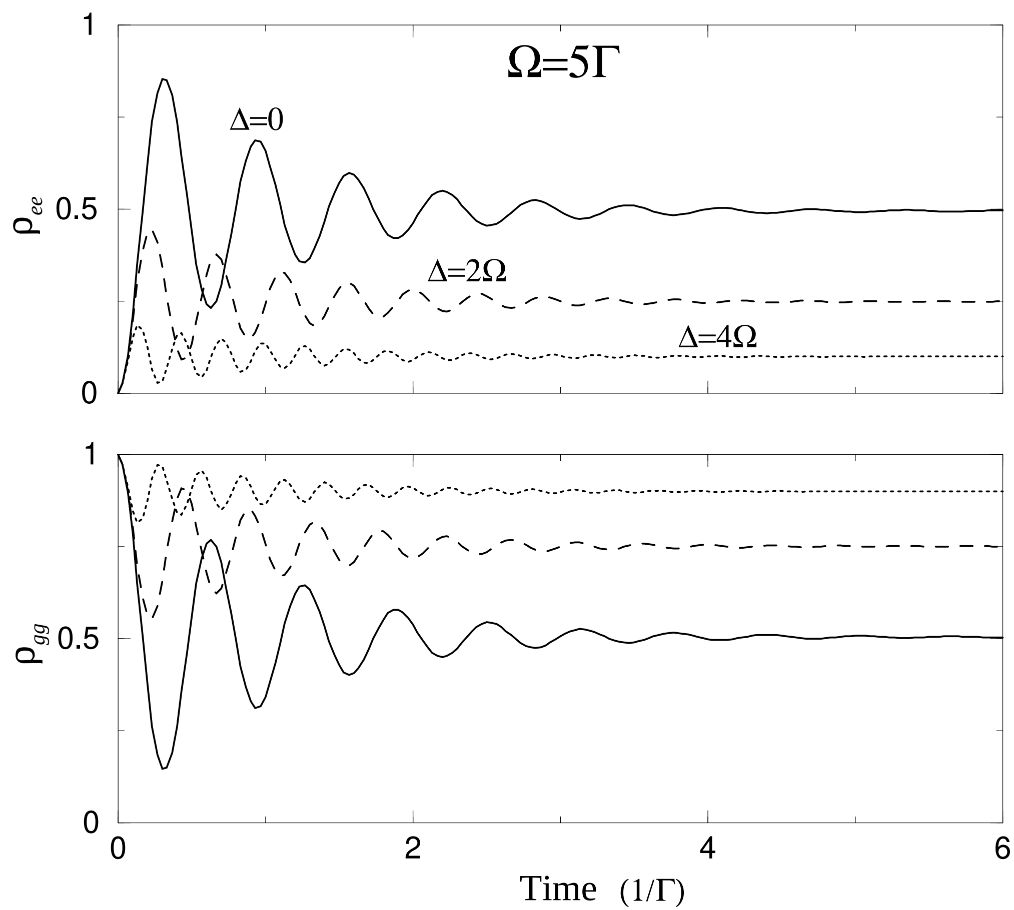

# Eigenstates (dressed states)

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**Interaction Hamiltonian** (rotating frame $\omega$)

$$\mathcal{H}_{\rm int} = -\hbar\Delta\,|e\rangle\langle e| - \hbar\Omega(\,|e\rangle\langle g| + |g\rangle\langle e|\,) = -\hbar\begin{bmatrix} 0 & \Omega \\ \Omega & \Delta \end{bmatrix} \qquad \{\,|g\rangle,\,|e\rangle\}$$

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Eigenvalue problem $\mathcal{H}_{\text{int}}\,|\Psi\rangle = \hbar\lambda\,|\Psi\rangle$

$$\Rightarrow \quad \lambda_\pm = \frac{\Delta \pm \sqrt{\Delta^2+4\Omega^2}}{2}$$

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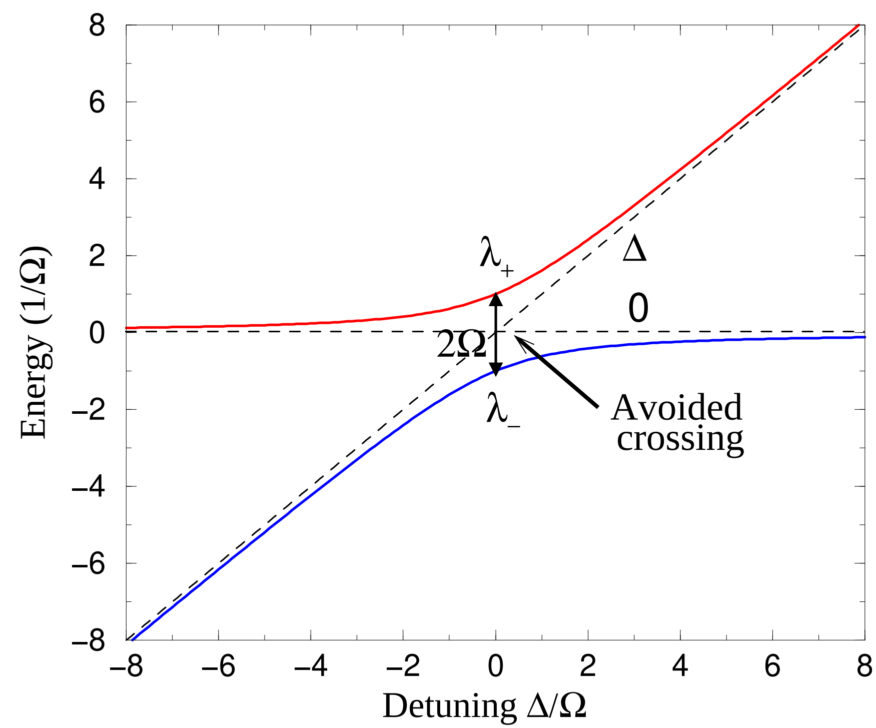

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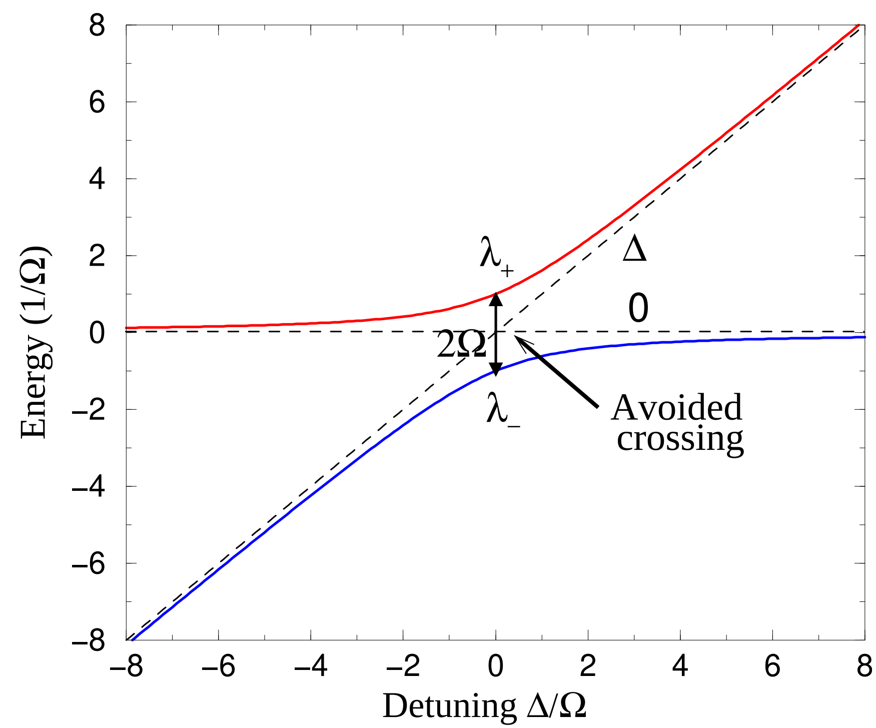

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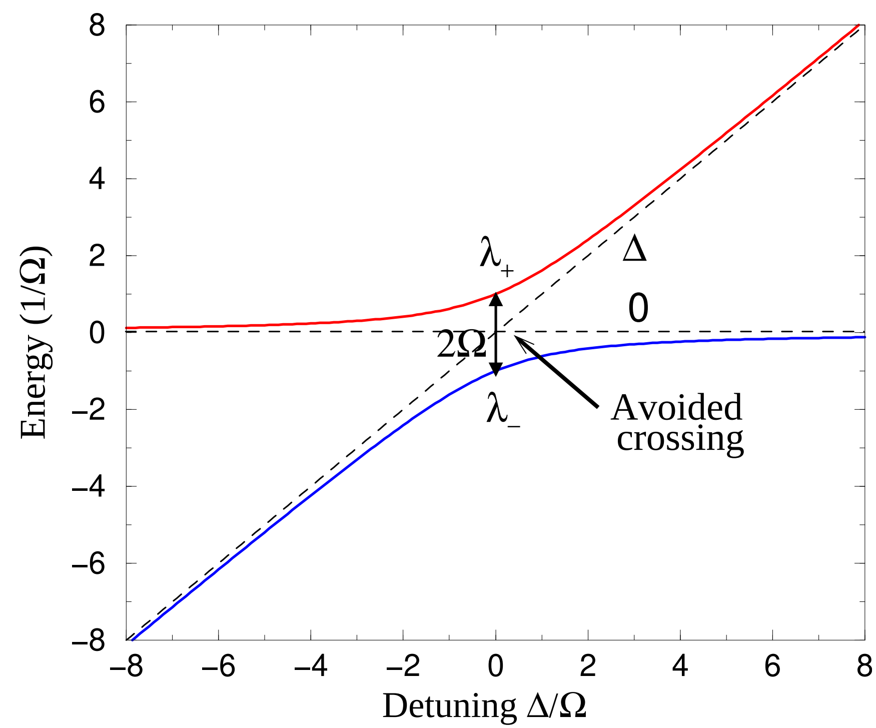
- $\Delta \gg |\Omega| \;\; (\Delta > 0) \Rightarrow$
  $\lambda_+ \rightarrow \Delta \qquad |+\rangle \rightarrow |e\rangle$
  $\lambda_- \rightarrow 0 \qquad |-\rangle \rightarrow |g\rangle$



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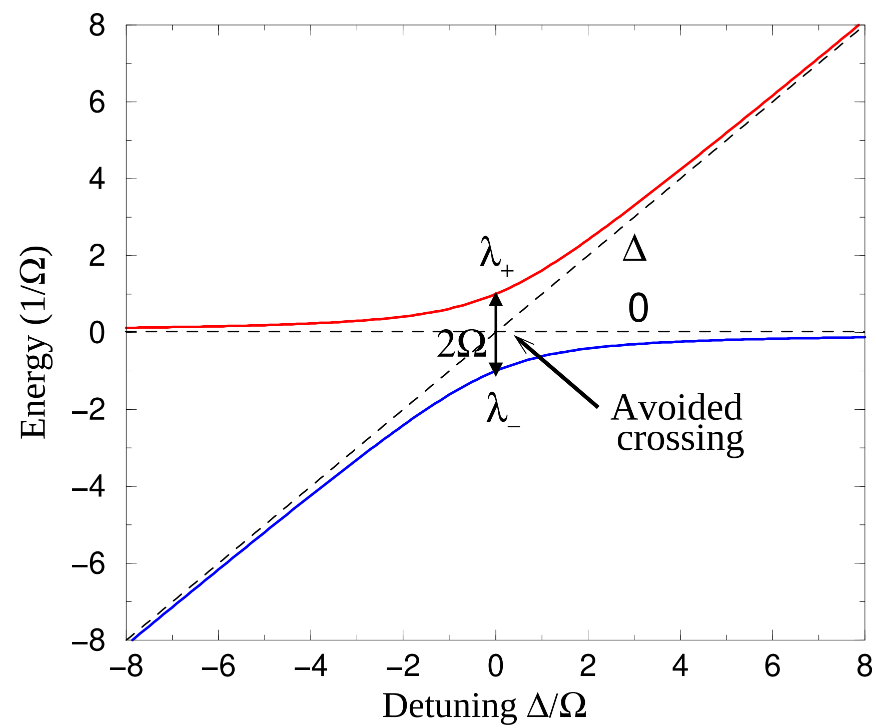
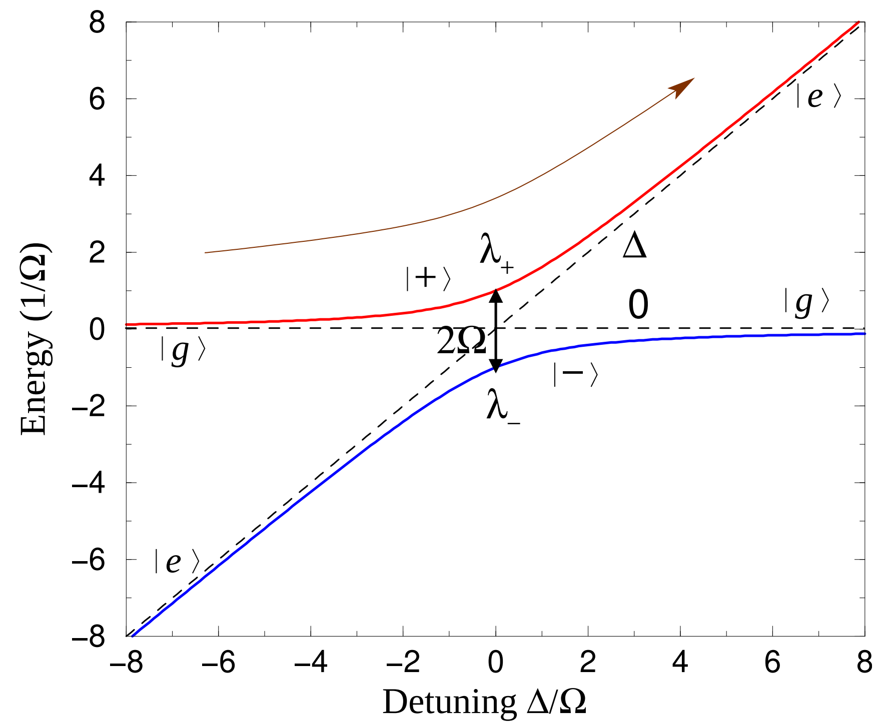


# Adiabatic & Non-Adiabatic Transitions

## Time-dependent detuning

$$\Delta(t) = at \quad (a > 0) \qquad [\omega(t) \text{ or } \omega_{eg}(t)]$$



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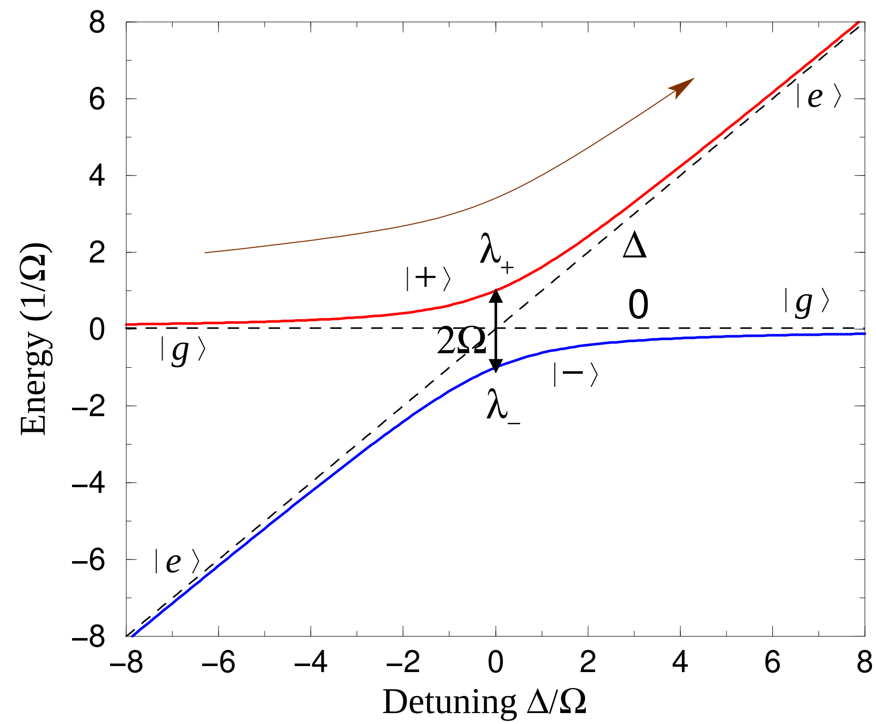
Initial state $|\Psi(-\infty)\rangle = |g\rangle \simeq |+\rangle$

Final state $|\Psi(\infty)\rangle = |e\rangle \simeq |+\rangle$ (?)



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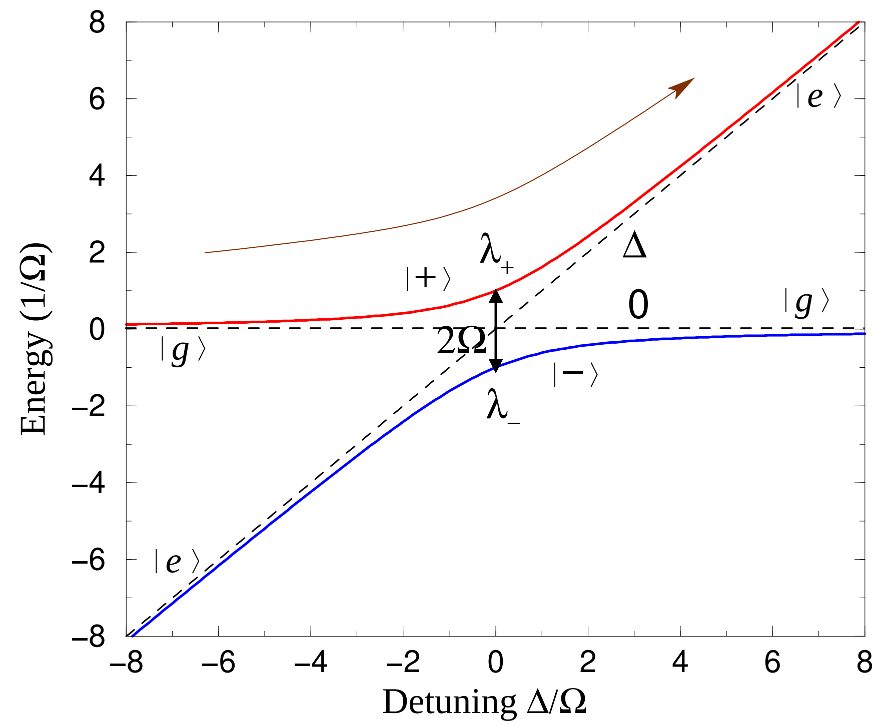


Non-adiabatic $|+\rangle \to |-\rangle$ transition probability (Landau-Zener formula)

$$P_{\mathrm{tr}} = e^{-2\pi\Gamma} \qquad \Gamma = \frac{\Omega^2}{\frac{\partial}{\partial t}|\lambda_+ - \lambda_i|} \sim \frac{\Omega^2}{a}$$

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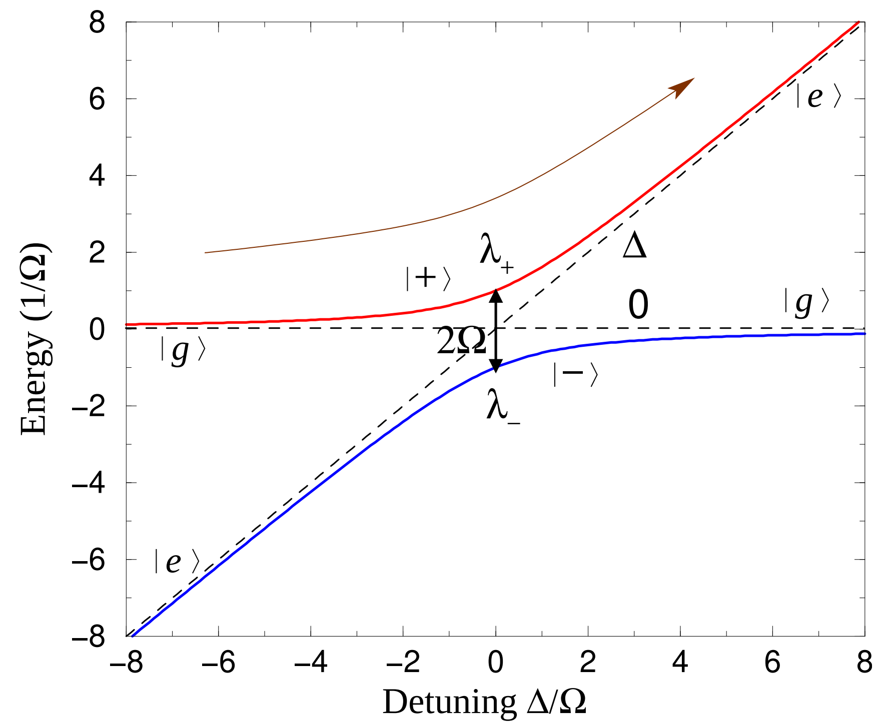


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$\Omega^2 \gg a \quad \Rightarrow \quad \Gamma \gg 1 \;\&\; P_{\rm tr} \ll 1$: Adiabatic following $|\Psi(t)\rangle = |+\rangle \; \forall \, t$

**Adiabatic transfer** $|g\rangle \to |e\rangle$

# Problems & Questions

- A resonant ($\Delta = 0$) coherent field with Rabi frequency $\Omega$ can prepare an equally weighted superposition of atomic states $|g, e\rangle$ for $\pi/2$-pulse $2\int dt\Omega(t) = 2\Omega T_{\pi/2} = \pi/2$ (assuming square pulse).
  For a non-resonant field $|\Delta| \leq 2\Omega$, what would be the pulse area/duration for preparing $|c_{g,e}| = \frac{1}{\sqrt{2}}$? What would be the relative phases for the amplitudes?

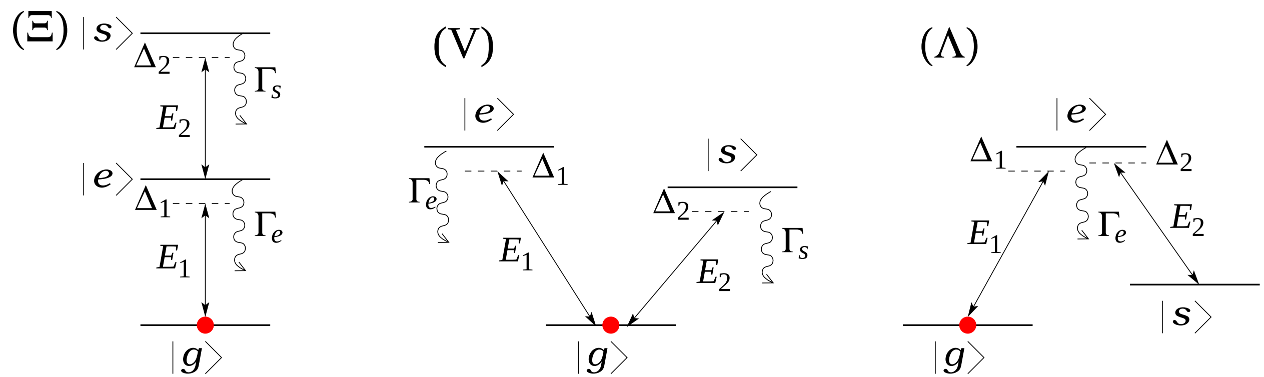
- In a Landau-Zener process of some duration $T = 1/v$, governed by the Schrödinder eq. $iv\partial_\tau |\Psi(\tau)\rangle = \mathcal{H}(\tau) |\Psi(\tau)\rangle$ ($\tau \equiv vt$), show that the non-adiabatic couling between the instantanous eigenstates $|\Psi_\pm\rangle$ [$\mathcal{H}(\tau) |\Psi_\pm\rangle = \lambda_\pm |\Psi_\pm\rangle$) is

$$-iv\langle\Psi_+| \partial_\tau |\Psi_-\rangle = -i\frac{v}{\lambda_+ - \lambda_-}\langle\Psi_+| \partial_\tau \mathcal{H}(\tau) |\Psi_-\rangle$$

# III. Three-level atom

# Atom-Fields Interactions

$$\mathcal{H} = \mathcal{H}^{\mathrm{A}} + \mathcal{V}^{\mathrm{AF}}(t)$$



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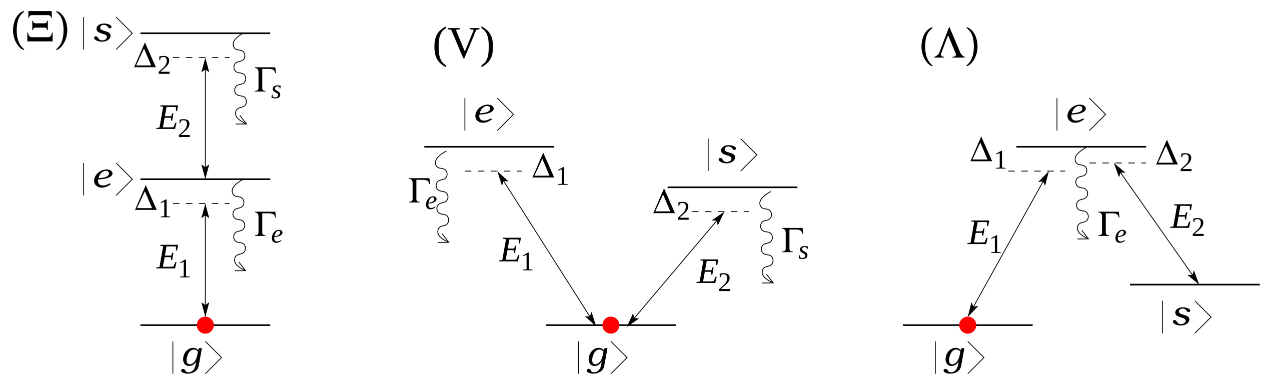


$$\mathcal{H}^{\rm A} = \hbar\omega_g \,|g\rangle\langle g| \,+ \hbar\omega_e \,|e\rangle\langle e| \,+ \hbar\omega_s \,|s\rangle\langle s|$$

$$\mathcal{V}^{\rm AF}(t) = -\boldsymbol{\wp} \cdot [\boldsymbol{E}_1(t) + \boldsymbol{E}_2(t)] \qquad \wp_{\mu\nu} \equiv \langle\mu|\, \boldsymbol{\wp} \cdot \boldsymbol{e}_j \,|\nu\rangle$$

$$= -\wp_{eg}|e\rangle\langle g|\mathcal{E}_1 e^{-i\omega_1 t} - \wp_{es(gs)}|e(g)\rangle\langle s|\mathcal{E}_2 e^{-i\omega_2 t} + {\rm H.c.}$$

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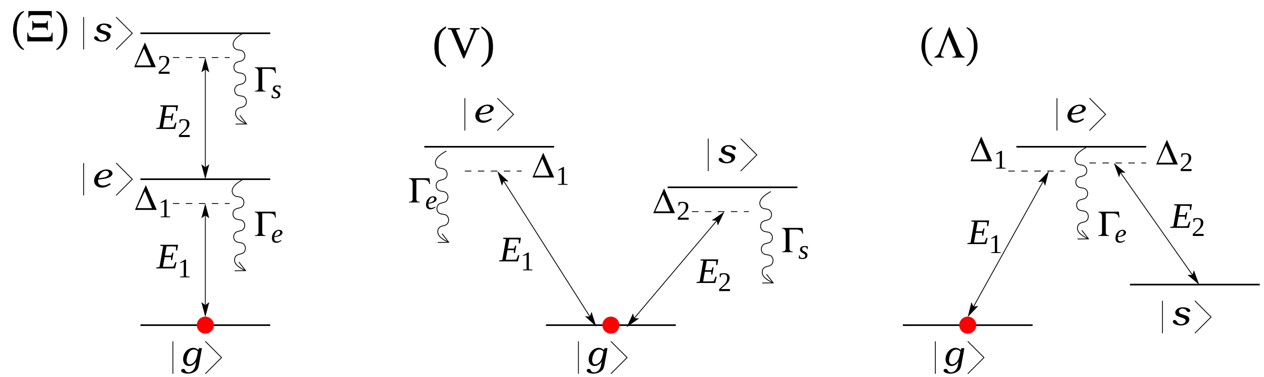


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## Interaction Hamiltonian

(rotating frame $\omega_{1,2}$)

$$\mathcal{H}_{\Xi,\Lambda} = -\hbar\begin{bmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & \Delta_1 & \Omega_2 \\ 0 & \Omega_2 & \Delta_1 \pm \Delta_2 \end{bmatrix} \qquad \mathcal{H}_V = -\hbar\begin{bmatrix} 0 & \Omega_1 & \Omega_2 \\ \Omega_1 & \Delta_1 & 0 \\ \Omega_2 & 0 & \Delta_2 \end{bmatrix}$$

$$\{\,|g\rangle,\,|e\rangle,\,|s\rangle\} \qquad \Delta_{1,2} = \omega_{1,2} - \omega_{\mu\nu} \qquad \Omega_{1,2} = \frac{\wp_{\mu\nu}}{\hbar}\mathcal{E}_{1,2}$$

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$$|\Psi(t)\rangle = c_g(t)\,|g\rangle + c_e(t)\,|e\rangle + c_s(t)\,|s\rangle \qquad \frac{\partial}{\partial t}\,|\Psi(t)\rangle = -\frac{i}{\hbar}\mathcal{H}_\Lambda\,|\Psi(t)\rangle \quad \Rightarrow$$

$$\frac{\partial}{\partial t}c_g = i\Omega_1 c_e$$

$$\frac{\partial}{\partial t}c_e = (i\Delta_1 - \gamma_e)c_e + i\Omega_1 c_g + i\Omega_2 c_s$$

$$\frac{\partial}{\partial t}c_s = i(\Delta_1 - \Delta_2)c_s + i\Omega_2 c_e$$

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$$\frac{\partial}{\partial t}c_s = i(\Delta_1 - \Delta_2)c_s + i\Omega_2 c_e$$

**Resonant interaction** $\Delta_1 = \Delta_2 = 0 \quad \Omega_{1,2} = \sqrt{2}\Omega \gg \gamma_e \quad \Rightarrow$

$$c_g(t) = \cos^2(\Omega t)$$

$$c_e(t) = i\sqrt{2}\sin(\Omega t)\cos(\Omega t)$$

$$c_s(t) = -\sin^2(\Omega t)$$

# Rabi Oscillations

$$|\Psi(t)\rangle = c_g(t)\,|g\rangle + c_e(t)\,|e\rangle + c_s(t)\,|s\rangle \qquad \frac{\partial}{\partial t}\,|\Psi(t)\rangle = -\frac{i}{\hbar}\mathcal{H}_\Lambda\,|\Psi(t)\rangle \quad \Rightarrow$$

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**Non-resonant interaction** $\Delta_1 \gg \Omega_{1,2}, \gamma_e \quad \Rightarrow$

$$c_e(t) = i\int_0^t e^{(i\Delta_1-\gamma_e)(t-t')}\big[\Omega_1 c_g(t') + \Omega_2 c_s(t')\big]dt'$$

$$\simeq i\big[\Omega_1 c_g(t) + \Omega_2 c_s(t)\big]\int_0^t e^{(i\Delta_1-\gamma_e)(t-t')}dt' = \frac{\Omega_1 c_g + \Omega_2 c_s}{\Delta_1 + i\gamma_e} \qquad (\gamma_e t \gg 1)$$

# Rabi Oscillations

FORTH IESL

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$\Rightarrow$ Two-photon (Raman) transition $|g\rangle \to |s\rangle$

$$\frac{\partial}{\partial t}c_g = -(iS_g + \gamma_g)c_g + i\Omega_{\text{eff}}\, c_e$$

$\boxed{\Omega_{\text{eff}} = \frac{\Omega_1\Omega_2}{\Delta_1}}$ eff. (2-phot.) Rabi fr.

$$\frac{\partial}{\partial t}c_s = -\left[i(S_s - \Delta_1 + \Delta_2) + \gamma_s\right]c_s + i\Omega_{\text{eff}}\, c_g$$

Stark shifts of $|g,e\rangle$: $S_{g,s} = \frac{|\Omega_{1,2}|^2}{\Delta_1}$; decays $\gamma_{g,s} = \gamma_e\frac{S_{g,s}}{\Delta_1} \to 0$

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$$\frac{\partial}{\partial t} c_g = i\Omega_{\text{eff}}\, c_s$$

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**Solution** $c_g(0) = 1, \;\; c_s(0) = 0$

$$c_g(t) = \cos(\bar{\Omega}_{\text{eff}} t) - i\frac{\Delta_{\text{eff}}}{2\bar{\Omega}_{\text{eff}}} \sin(\bar{\Omega}_{\text{eff}} t)$$

$$c_s(t) = i\frac{\Omega_{\text{eff}}}{\bar{\Omega}_{\text{eff}}} \sin(\bar{\Omega}_{\text{eff}} t)$$

$$\bar{\Omega}_{\text{eff}} \equiv \sqrt{\Omega_{\text{eff}}^2 + (\Delta_{\text{eff}}/2)^2}$$

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**Two-photon resonance** $\Delta_{\text{eff}} = 0 \quad \Rightarrow$

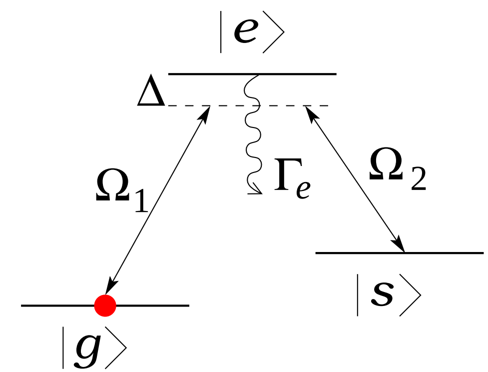
$$c_g(t) = \cos(\Omega_{\text{eff}} t) \qquad c_s(t) = i \sin(\Omega_{\text{eff}} t)$$

Two-photon Rabi oscillations

# Eigenstates (dressed states)

**Two-photon resonance** $\Delta_1 = \Delta_2 \equiv \Delta$

$$\mathcal{H}_\Lambda = -\hbar \begin{bmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & \Delta & \Omega_2 \\ 0 & \Omega_2 & 0 \end{bmatrix}$$



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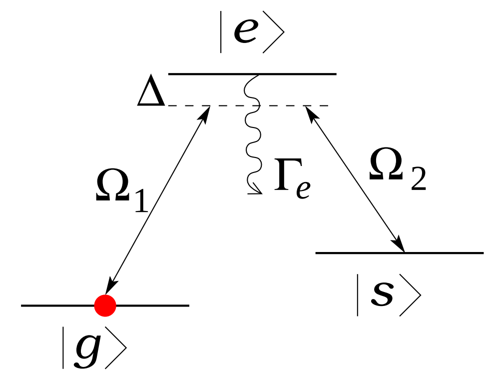


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Eigenvalue problem $\mathcal{H}_\Lambda |\Psi\rangle = \hbar\lambda |\Psi\rangle$

$\Rightarrow \quad \lambda_0 = 0 \qquad \lambda_\pm = -(\Delta/2) \pm \bar{\Omega} \qquad \bar{\Omega} = \sqrt{\Omega_1^2 + \Omega_2^2 + (\Delta/2)^2}$

$$|D\rangle = \frac{1}{\sqrt{N_0}}[\Omega_2 |g\rangle - \Omega_1 |s\rangle]$$

$$|B_\pm\rangle = \frac{1}{\sqrt{N_\pm}}[\Omega_1 |g\rangle - \lambda_\pm |e\rangle + \Omega_2 |s\rangle]$$

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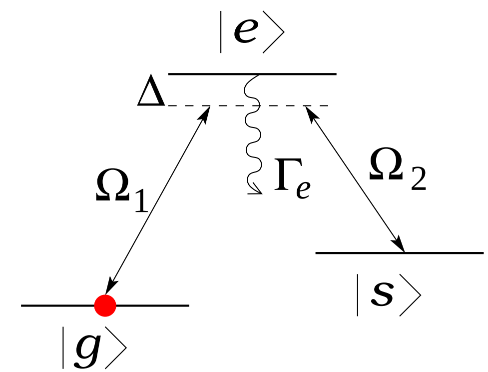


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**Dark (coherent population trapping – CPT) state**

$$|D\rangle = \cos\Theta |g\rangle - \sin\Theta |s\rangle \qquad \tan\Theta = \frac{\Omega_1}{\Omega_2}$$ mixing angle

$$\Rightarrow \quad |c_g|^2 = \cos^2\Theta = \frac{\Omega_2^2}{\Omega_1^2+\Omega_2^2} \qquad |c_e|^2 = 0 \qquad |c_s|^2 = \sin^2\Theta = \frac{\Omega_1^2}{\Omega_1^2+\Omega_2^2}$$

# Stimulated Raman Adiabatic Passage

## Time-dependent Rabi frequencies $(\Delta = 0)$

- $\Omega_2(t_\mathrm{i}) \gg \Omega_1(t_\mathrm{i}) \quad [\Theta = 0]$

  $\Rightarrow \ |\Psi(t_\mathrm{i})\rangle = |D\rangle = |g\rangle$

- $\Omega_2(t_\mathrm{f}) \ll \Omega_1(t_\mathrm{f}) \quad [\Theta = \frac{\pi}{2}]$

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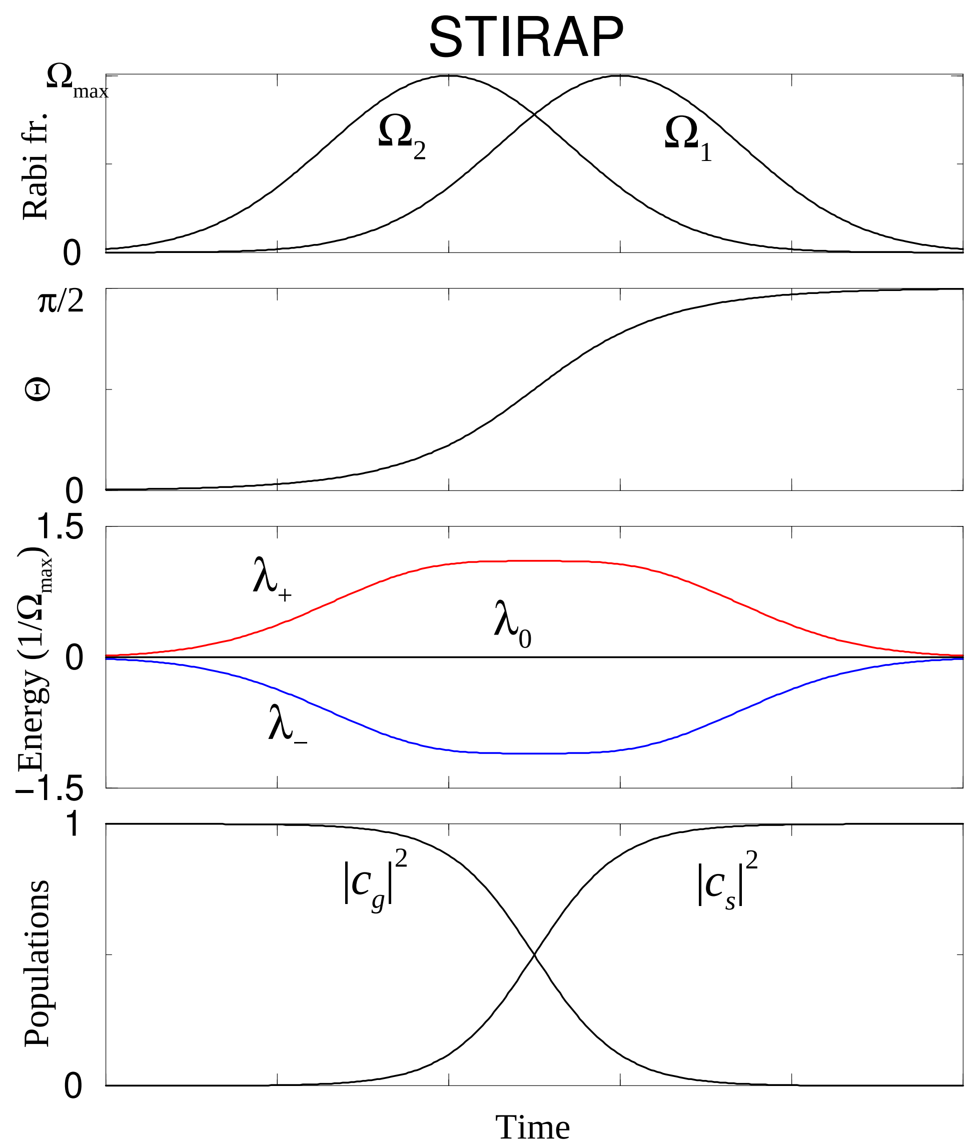

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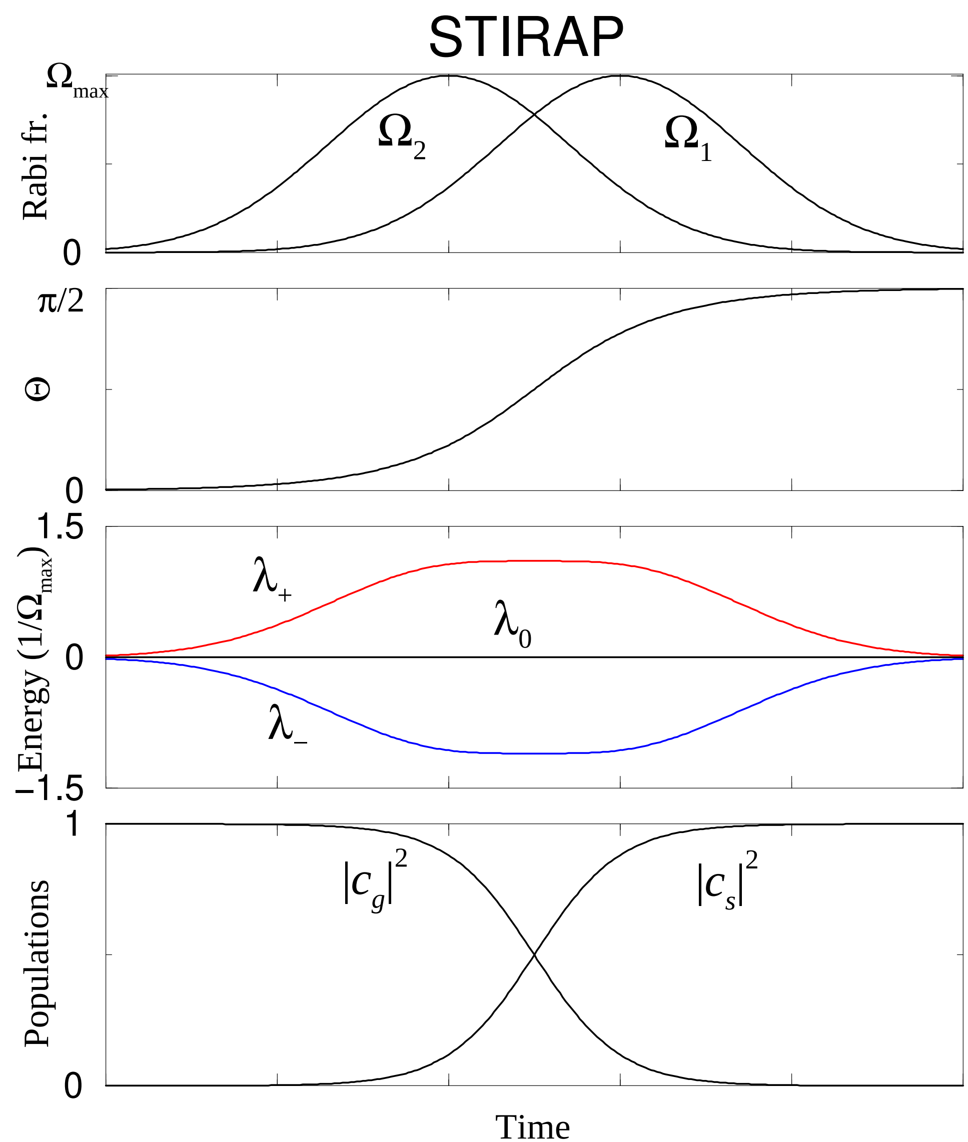


## Adiabatic following condition

$$|\dot{\Theta}| = \left| \frac{\dot{\Omega}_1 \Omega_2 - \Omega_1 \dot{\Omega}_2}{\Omega_1^2 + \Omega_2^2} \right| \ll |\lambda_\pm - \lambda_0|$$

Gaussian pulses of duration $T$

$$\Rightarrow \quad \Omega_\mathrm{max} T \gtrsim 10$$

# Problems & Questions

- Starting with the Hamiltonian $\mathcal{H}_{\Xi}$ for a three-level atoms with $\Xi$ configuration of levels, including the decay of levels $|e\rangle$ and $|s\rangle$, derive the coupled differential equations for the amplitudes $c_g$ and $c_s$ under the conditions $|\Delta_1 + \Delta_2| \ll |\Delta_{1,2}|$ and $\sqrt{\Delta_{1,2}^2 + \gamma_e} \gg \Omega_{1,2}$.
- Under what conditions, the ac Stark shifts $S_{g,s}$ of levels $|g, s\rangle$ in a $\Lambda$ and $\Xi$ systems are equal in magnitude and sign?
- In a V-system, can we perform STIRAP between levels $|e\rangle$ and $|s\rangle$ having large decay rates $\Gamma_{e,s}$?

# Further Reading


- L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, 1975)
- C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (Wiley, 1992)
- M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997)
- P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer, 1999)
- P. Lambropoulos and D. Petrosyan, *Fundamentals of Quantum Optics and Quantum Information* (Springer, 2007)

- L. Landau, Phys. Z. **2**, 46 (1932); C. Zener, Proc. R. Soc. London A **137**, 696 (1932)
- E. Arimondo, *Coherent population trapping in laser spectroscopy*, Prog. Opt. **35**, 257 (1996)
- K. Bergmann, H. Theuer and B.W. Shore, *Coherent population transfer among quantum states of atoms and molecules*, Rev. Mod. Phys. **70**, 1003 (1998); Rev. Mod. Phys. **89**, 015006 (2017)


http://gate.iesl.forth.gr/~dap/lnotes.html