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Acoustic Waves in Random Media

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Abstract. – Motivated by recent experimental results we test how well various extensions of the well-known coherent potential approximation (CPA) determine the characteristics of the acoustic wave propagation through a random composite consisting of glass spheres in water. In spite of the approximate character of the methods, our results seem to be in reasonable agreement with recent experimental data.

The coherent potential approximation (CPA) method widely used for electrons in disordered solids has been extended in recent years to the problem of classical wave (CW) propagation through random systems [1, 2, 3]. The efforts to extend the CPA to the CW case gave rise to more than one versions of the method depending on the nature of the CW, the structure of the system, and the quantity to be calculated. Various versions of the CPA have been applied to simple scalar waves (SSW) and to electromagnetic waves (EMW) with results in reasonable agreement with the existing experimental data [1, 2, 3].

On the other hand acoustic (AC) and elastic (EL) waves have received less attention. Their study has been restricted mostly to periodic systems [4] or random systems in the low concentration or long wavelength limit [1, 5]. The main reasons for this scarcity of calculations are the theoretical difficulties of the problem combined with the absence of reliable experimental data. The theoretical difficulties stem from *i*) the full vector character of the ELW giving rise to scattering induced mode conversion; *ii*) the more than one parameter which characterize a homogeneous elastic medium (density and velocities) which require a larger number of equations for their calculation. Thus one has to find a set of equations which would produce optimum results without creating serious computational problems.

On the experimental side the difficulty is in the interpretation of velocity measurements in the very interesting and poorly understood regime of resonant multiple scattering. In this regime the study of the EMW and SSW has shown that the energy transport velocity, v_E, *i.e.* the velocity entering the diffusion constant, D ($= v_E l/3$), is the appropriate velocity characterizing the propagation [6, 7, 8]. The velocity v_E, as a function of frequency, was found

to exhibit pronounced dips near the single scattering resonances at low concentrations while as the concentration increases its frequency dependence becomes smoother [3, 6, 7, 8, 9, 10]. The dips at low concentrations are attributed to the delay of the wave inside the scatterers near the resonances while the smoothness as the concentration increases is due to the fact that the scattering material now provides an additional path for propagation because of the coherent hopping of the wave from scatterer to nearby scatterers.

Recently, Page *et. al.* measured for the first time, to the best of our knowledge, the diffusion coefficient, the mean free paths (determining thus the transport velocity) as well as the phase and group velocity for ACW propagating through a suspension of glass beads of radius a immersed in water *in the strong scattering regime* [11, 12]. They determined velocities far from the velocity of glass although the volume fraction of the glass beads (63%) was near the close-packing. The present work is a brief presentation of calculational results motivated by their data [11, 12], which thus allow us to test various extensions of the CPA in the particular case examined by Page *et. al.*

We consider a random composite consisting of glass spheres (material i (in) of volume fraction $f_s = 63\%$) immersed in water host (material o (out)). We calculate the phase and energy velocity, the scattering mean free path and the localization parameter in the composite by the use of the simple CPA (S-CPA), the coated CPA (C-CPA) [2], and two versions of an energy based CPA (E-CPA1 and E-CPA2, see below) [9].

The basic idea of the S-CPA and C-CPA is the replacement of the random medium by a homogeneous effective medium with a complex propagation vector q_e which is calculated self-consistently [10, 13] by requiring that the scattering arising from the local substitution of the effective medium by the actual medium should vanish on the average. Thus the effective vector q_e is calculated by :

$$p_1 f_1(\mathbf{q}_e, \mathbf{q}_e) + p_2 f_2(\mathbf{q}_e, \mathbf{q}_e) = 0 \tag{1}$$

where $f_i(\mathbf{q}_e, \mathbf{q}_e)$ $(i = 1, 2)$ is the forward scattering amplitude for an incident plane wave scattered by a scattering unit of the type i embedded in the effective medium with probability p_i.

Within the S-CPA the scattering unit of the type 1 (2) is regarded as a glass (water) sphere of radius a embedded in the effective medium with probability $p_1 = f_s$ $(p_2 = 1 - f_s)$. Within the C-CPA the scattering unit of the type 1 is regarded as a coated sphere (the actual glass sphere coated with a water coating) of external radius r_1 and the scattering unit of the type 2 as a simple water sphere of radius r_2 (this choice reproduces the fact that the two materials are not topologically equivalent). The radii of the two scattering units and the corresponding probabilities throughout the present work are chosen either in the same way as in ref. [2] or by the relations : $p_1 = p_2 = 1/2$, $r_1 = 0.9386a/f_s^{1/3}$, $r_2 = 0.5572a/f_s^{1/3}$ with practically the same results in all cases which have been examined.

From q_e one can calculate immediately the scattering mean free path $l_s = 0.5/Im(q_e)$, the phase velocity $c = \omega/Re(q_e)$ and the localization parameter $k_e l_s$.

The self-consistency condition for the determination of the real effective propagation constant, k_m, within the energy based CPA is that the total energy stored in a scattering unit embedded in the homogeneous medium to be equal with the energy stored in a region of the homogeneous medium of the same volume as this scattering unit [9]. The energies are calculated by considering an incident plane wave. The basic scattering unit is regarded either as a single coated sphere of radius $r_c^3 = a^3/f_s$ and concentration $n_1 = f_s/V_a$, where $V_a = 4\pi a^3/3$ (E-CPA1) or a coated sphere (of radius r_1 and concentration $n_1 = f_s/V_a$) *and* a simple sphere (of radius r_2 and concentration $n_2 = (1 - n_1 V_1)/V_2$) (E-CPA2). $V_j = 4\pi r_j^3/3$, $j = 1, 2$ and

r_1, r_2 are as in the C-CPA.

The complex effective propagation vector, q_e, within either E-CPA1 or E-CPA2 is determined by :

$$q_e^2 = k_m^2 - \Sigma(\omega) \tag{2}$$

where the self-energy $\Sigma(\omega)$ is obtained by using its low concentration expression [10] :

$$\Sigma(\omega) = -4\pi \sum_i n_i f_i(\mathbf{k}_m, \mathbf{k}_m) \tag{3}$$

In eq.(3), $f_i(\mathbf{k}_m, \mathbf{k}_m)$ is the forward scattering amplitude for an incident plane wave scattered by a scattering unit of the type i embedded in the homogeneous medium k_m.

Within the E-CPA1 or E-CPA2 approach the transport velocity, v_E, is calculated by [6, 7, 8, 3, 10] :

$$v_E = \frac{\omega}{k_m^2} \frac{\sqrt{k_m^2 - Re(\Sigma)}}{1 + \sum_i n_i \delta_i} \tag{4}$$

where the quantity δ_i is given by [6, 7, 8] :

$$\delta_i = 4\pi Re[\frac{\partial f_i(\mathbf{k}_m, \mathbf{k}_m)}{\partial k_m^2}] + \int d\Omega \frac{d\sigma^i}{d\Omega} \frac{\partial \phi^i_{\mathbf{k}_m \mathbf{k}'_m}}{\partial k_m} \tag{5}$$

$\phi^i_{\mathbf{k}_m \mathbf{k}'_m}$ is the phase of $f_i(\mathbf{k}_m, \mathbf{k}'_m)$ and $\frac{d\sigma^i}{d\Omega} = |f_i(\mathbf{k}_m, \mathbf{k}'_m)|^2$.

In contrast to the SS and the EM waves where only one material parameter, namely the velocity, defines the effective medium, for ACW one needs two parameters (*e.g.* velocity and density). Thus one more self-consistency condition besides eq.(1) is needed. For the S-CPA and C-CPA we attempted to put the average scattering amplitude in a direction other than the forward equal to zero as well. In several cases this led us to convergence difficulties and/or unphysical multiple solutions. We employed also an approximate expression for the effective density (instead of a self-consistent one) appropriate for the long wavelength limit [5] as well as the simple average $\bar{\rho} = f_s \rho_i + (1 - f_s)\rho_o$. In the cases where the two self-consistency conditions provide uniquely convergent solutions, the results were very close to those obtained by using eq.(1) and the approximate expression of ref. [5] for the effective density. In the E-CPA the density of the effective medium, ρ_m, was replaced either by the average density or by the long wavelength limit effective density [5] with practically the same results.

Recently, success in predicting and understanding the basic features of the ACW and ELW propagation in periodic systems was achieved by the use of the single scattering analysis [14]. Below, we apply this analysis in order to provide an explanation for of our CPA's results. We present first the total scattering cross section, σ, for a plane ACW scattered by a glass sphere immersed in water (fig.1 - solid line). One can see that the scattering cross section is not very large. We show also the scattering cross section, σ_h, by a rigid (hard) sphere immersed in the water (dashed line), and the cross section, σ_{sb}, calculated by subtracting from the glass sphere scattering amplitude the rigid sphere scattering amplitude (dotted line) [14, 15]. σ_{sb} is clearly lower than either σ or σ_h, especially at the (rather weak) resonances, which means that the wave does not penetrate appreciably within the glass sphere. This was verified directly by calculating the energy density as a function of the distance from the center of the glass sphere.

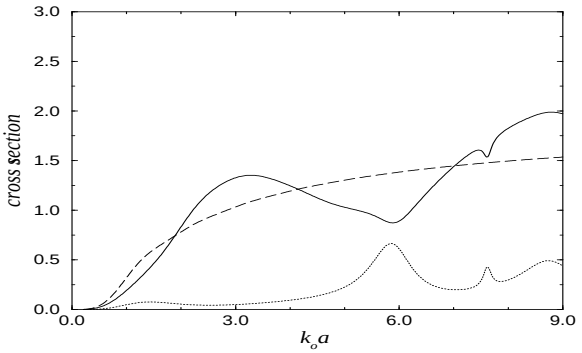

Fig. 1. – Total dimensionless cross section ($\sigma/\pi a^2$) vs $k_o a = \omega a/c_o$ for a glass sphere (solid line) and for a rigid sphere (dashed line) both of radius a embedded in a water host. The dotted line represents the cross-section calculated by subtracting from the glass sphere scattering amplitude the rigid sphere scattering amplitude; ω is the frequency and c_o the wave velocity in the water.

As a result we expect the propagation velocity to be close to that of water. Furthermore, its frequency variation is expected to be weak as a result of the rather smooth σ vs ω.

In fig.2a we show our results for v_E according to the E-CPA2 together with the experimental point [11]. The discrepancy is of the order of 10% or less. Note that the values of v_E are very close to c_o and they do not exhibit any strong variation with frequency. Both of these features are consistent with our expectations based on the previous analysis of the single scattering. Furthermore, the weak dips of v_E at $k_o a \approx 6$ and 7.7 can possibly be attributed to the delay of wave propagation due to its increased penetration inside the spheres as evidenced by the corresponding peaks in σ_{sb}. We have calculated v_E at lower concentrations as well and we found that the form of v_E vs ω is similar to that of fig.2a reinforcing thus our interpretation.

In fig.2b we present results for the phase velocity, c_{ph}, according to S-CPA, C-CPA and E-CPA2. The agreement with the experimental results, especially for the C-CPA, is very good to excellent with the exception of the frequency region arount $k_o a \approx 2.5$ ($k_o = \omega/c_o$). We think that this discrepancy (which can be reduced slightly if we average over the fluctuation in the size of the glass spheres) is due to the *increase* of the multiple scattering in this frequency region. This increase is due (i) to the resonance in the single scattering and (ii) to the matching of the wavelength $\lambda_o = 2\pi/k_o$ with the nearest neighbor separation $d \approx 2a$, at around $k_o a \approx 2.5$. This matching makes the multiple scattering not only stronger but more coherent as well, making thus more difficult for the CPA to describe the effect.

In fig.3 we show the scattering mean free path as a function of frequency calculated by employing the same approximations for the effective density and the same configurations as in the case of fig.2. The triangles and the circles indicate Page's *et. al.* experimental result [12] while the dotted-dashed line shows the mean free path calculated by using the low concentration expression $l_s = 1/n\sigma$ (n is the number density of the scatterers in the actual material and σ the single glass sphere cross section; it is remarkable that this simple formula produces results not so different from the experimental ones in this very high concentration system).

We can see that the mean free path which is calculated within the C-CPA approximates better the experimental data than the S-CPA or the E-CPA2 mean free path. The discrepancy between C-CPA and experimental result at $k_o a \approx 2.5$, as well as, the frequency shift of the theoretical curves compared to the experimental data can been attributed again to the increase of the strength and the coherent nature of the multiple scattering discussed in connection with

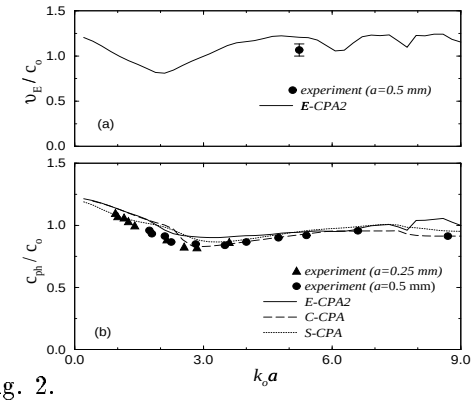

Fig. 2.

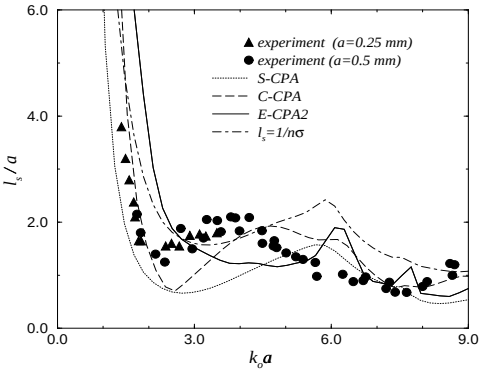

Fig. 3.

Fig. 2. – Energy transport velocity v_E (a), and phase velocity c_{ph} (b), versus the dimensionless frequency $k_o a = \omega a / c_o$ for glass spheres of volume fraction 63% randomly placed in water. The triangles and the circles indicate Page's *et. al.* experimental result for spheres of radius 0.25 mm and 0.5 mm respectively. The solid, dashed, and dotted lines indicate the velocities calculated within the E-CPA2, the C-CPA and the S-CPA respectively. c_o is the wave velocity in the water and a the sphere radius.

Fig. 3. – Scattering mean free path, l_s, (in units of the sphere radius a), versus the dimensionless frequency $k_o a = \omega a / c_o$ for glass spheres of volume fraction 63% randomly placed in water. The mean free path is calculated within the S-CPA (dotted line), the C-CPA (dashed line), the E-CPA2 (solid line) and by the low concentration expression $l_s = 1/n\sigma$ (dotted-dashed line). The triangles and the circles indicate Page's *et. al.* experimental result for spheres of radius 0.25 mm and 0.5 mm respectively. c_o is the wave velocity in the water.

fig.2. The discrepancies were reduced somehow when we took an average over slightly different sphere radii.

Concerning the differences between the E-CPA2 result and the others, one has to notice that the self-energy $\Sigma(\omega)$ (from which the E-CPA2 mean free path was obtained (see eq.(2))) was calculated by eq.(3) which is clearly a low concentration expression. In contrast, the S-CPA and C-CPA do not employ any explicit low concentration approximation for obtaining either velocity or mean free path. The above mentioned discrepancy becomes more pronounced if one employs E-CPA1. We point out that all the methods tend, as expected, to the same low concentration limit for the mean free path, $l_s = 1/n\sigma$.

Finally, we mention that the calculated values of the localization parameter $k_e l_s$ are not close to the localization threshold $(k_e l_s)_c = 0.84$ [16] (or $(k_e l_s)_c = 0.91$ [10]) (the minimum calculated value of $k_e l_s$ as the frequency varies is 1.98, *i.e.* far from the critical region). This implies the validity of the diffusion approximation in describing the energy propagation even at these high concentration regimes.

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